A LOCAL FAMILIES INDEX FORMULA FOR $\overline{\partial}$-OPERATORS ON PUNCTURED RIEMANN SURFACES

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Abstract. Using heat kernel methods developed by Vaillant, a local index formula is obtained for families of $\overline{\partial}$-operators on the Teichmüller universal curve of Riemann surfaces of genus g with n punctures. The formula also holds on the moduli space $\mathcal{M}_{g,n}$ in the sense of orbifolds where it can be written in terms of Mumford-Morita-Miller classes. The degree two part of the formula gives the curvature of the corresponding determinant line bundle equipped with the Quillen connection, a result originally obtained by Takhtajan and Zograf.

Contents

Introduction

Let X be a smooth even dimensional oriented compact manifold with boundary $\partial X \neq \emptyset$. Assume that the boundary is the total space of a fibration

$$\begin{array}{ccc} Z & \text{---} & \partial X \\ & & \downarrow \phi \\ & & Y \end{array} \tag{1}$$

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where Y and Z are compact oriented manifolds, Y being the base and Z being a typical fibre. Let $x \in \mathcal{C}^\infty(X)$ be a boundary defining function for X and

$$c : \partial X \times [0, \epsilon)_x \to N \subset X \tag{2}$$

a corresponding collar neighborhood of ∂X in X. Let g_{hc} be a metric on $X \setminus \partial X$ which takes the form

$$c^* g_{\mathrm{hc}} = \frac{dx}{x^2} + \phi^* g_Y + x^2 g_Z \tag{3}$$

in the collar neighborhood (2), where g_Z is a metric for the vertical tangent bundle of (1) and g_Y is a metric on the base Y which is lifted to ∂X using a choice of connection for the fibration (1). Such a metric is called a product fibred hyperbolic cusp metric (product d-metric in the terminology of [31]). If the manifolds X, Y and Z are spin, one can construct a Dirac operator associated to the metric g_{hc}. More generally, one can consider a Dirac type operator D constructed using the metric g_{hc} and a Clifford module $E \to X$ with a choice of Clifford connection.

In his thesis [31], Vaillant studied the index and the spectral theory of such operators. To do so, he introduced the conformally related operator xD and defined the vertical family by $D^V := xD|_{\partial X}$, which is a family of operators on ∂X parametrized by the base Y and acting on each fibre of (1). Assuming that the rank of $\ker D^V \to Y$ is constant so that it is a vector bundle over Y (constant rank assumption), Vaillant also introduced a horizontal operator

$$D^H : \mathcal{C}^\infty(Y; \ker D^V) \to \mathcal{C}^\infty(Y; \ker D^V) \tag{4}$$

which governs the continuous spectrum of D with bands of continuous spectrum starting at the eigenvalues of D^H and going out at infinity. In particular, the operator D is Fredholm if and only if D^H is invertible. In that case, Vaillant was able to obtain a formula for its index using heat kernel techniques and Getzler's rescaling along the lines of [22],

$$\operatorname{ind}(D) = \int_X \widehat{A}(R_{\mathrm{hc}}) \operatorname{Ch}(F^{E/S_{\mathrm{hc}}}) - \int_Y \widehat{A}(R_{g_Y}) \widehat{\eta}(D^V) - \frac{1}{2}\eta(D^H), \tag{5}$$

the first term being the usual Atiyah-Singer integral, $\widehat{\eta}(D^V)$ being the Bismut-Cheeger eta form of the vertical family and $\eta(D^H)$ being the eta invariant of D^H.

In [2], the authors, inspired by the work of Melrose and Piazza in [24] and [25], generalized the formula of Vaillant to families of Dirac type operators. Via the use of Fredholm perturbations, a notion intimately related to spectral sections, it was also possible to study situations where the constant rank assumption is not satisfied, allowing among other things to generalize the index theorem of Leichtnam, Mazzeo and Piazza [20].

The present paper, which is a sequel to [2], intends to put into use the index formula of [2] to study the following fundamental example arising in

Teichmüller theory. Assume that $2g+n \geq 3$ and let $T_{g,n}$ be the Teichmüller space of Riemann surfaces of genus g with n punctures. Let $p : \mathcal{T}_{g,n} \to T_{g,n}$ be the Teichmüller universal curve whose fibre above $[\Sigma] \in T_{g,n}$ is the corresponding Riemann surface Σ of genus g with n punctures. Let $T_v^{i,j}\mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ be the (i,j) vertical tangent bundle and let $\Lambda_v^{i,j}$ be its dual. In particular, $K_v := \Lambda_v^{1,0}$ restricts on each fibre Σ to the corresponding canonical line bundle $K_\Sigma := \Lambda_\Sigma^{1,0}$.

For each $\ell \in \mathbb{Z}$, one can associate a family of $\overline{\partial}$-operators

$$\overline{\partial}_\ell : \mathcal{C}^\infty(\mathcal{T}_{g,n}; K_v^\ell) \to \mathcal{C}^\infty(\mathcal{T}_{g,n}; \Lambda_v^{0,1} \otimes K_v^\ell) \tag{6}$$

acting fibre by fibre on $p : \mathcal{T}_{g,n} \to T_{g,n}$ and parametrized by the base $T_{g,n}$. By the uniformization theorem for Riemann surfaces, each fibre Σ of $p : \mathcal{T}_{g,n} \to T_{g,n}$ comes equipped with a hyperbolic metric g_Σ. Compactifying each fibre by a compact Riemann surface with boundary, these metrics can be seen as product hyperbolic cusp metrics, the fibration structure on the boundary being the collapsing map onto a point. With these metrics, the family $\overline{\partial}_\ell$ can be interpreted as a family of Dirac-type operators associated to a family of product hyperbolic cusp metrics. Using the criterion of Vaillant [31], one can check that each member of the family is Fredholm. The formula of [2] therefore applies.

As described in [34], the fibration $p : \mathcal{T}_{g,n} \to T_{g,n}$ is equipped with a canonical connection. This allows one to interpret the formula of [2] at the level of forms. In general, the eta forms involved in this formula are quite hard to compute. However, in this specific example, an explicit computation is possible using a result of Zhang [36], the vertical family being defined on a circle fibration. The **main result of this paper**, theorem 1, gives the following local family index formula,

$$\begin{aligned} \operatorname{Ch}(\operatorname{Ind}(\overline{\partial}_\ell)) &= \int_{\mathcal{T}_{g,n}/T_{g,n}} \operatorname{Ch}(T_v^{-\ell}(\mathcal{T}_{g,n}))\operatorname{Td}(T_v\mathcal{T}_{g,n}) + \frac{n}{2}\operatorname{sign}(\frac{1}{2}-\ell) \\ &- \sum_{i=1}^{n}\left(\frac{1}{2\tanh\left(\frac{e_i}{2}\right)} - \frac{1}{e_i}\right) - \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{N}{2}} d\int_0^\infty \operatorname{Str}\left(\frac{\partial \mathbb{A}_{D_\ell}^t}{\partial t} e^{-(\mathbb{A}_{D_\ell}^t)^2}\right) dt, \end{aligned} \tag{7}$$

where $\mathbb{A}_{D_\ell}^t$ is the Bismut superconnection and N is the number operator in $\Lambda T_{g,n}$. To define the form e_i, let $\mathcal{L}_i \to T_{g,n}$ be the complex line bundle which at $[\Sigma] \in T_{g,n}$ is given by the restriction of K_v at the ith puncture (marked point) of $\Sigma := p^{-1}([\Sigma])$. Then e_i is the Chern form of $\mathcal{L}_i$ as defined by Wolpert [35].

Since the Teichmüller space $T_{g,n}$ is contractible, formula (7) only contains cohomological information in its degree zero part. However, since it is local and each of its terms is invariant under the action of the Teichmüller modular group $\operatorname{Mod}_{g,n}$, formula (7) also holds on the moduli space $\mathcal{M}_{g,n} = T_{g,n}/\operatorname{Mod}_{g,n}$ in the sense of orbifolds, where the fibration $p : \mathcal{T}_{g,n} \to T_{g,n}$ is replaced by the forgetful map $\pi_{n+1} : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ and where it acquires a topological meaning in higher degrees (see Corollary

6.7). For instance, on the moduli space $\mathcal{M}_{g,n}$, the Chern form e_i represents the Miller class $\psi_i = c_1(\mathcal{L}_i)$, while the first term on the left-hand side of (7) represents a linear combination of the Mumford-Morita classes

$$\kappa_j = (\pi_{n+1})_*(c_1(\psi_{n+1}^{j+1})), \quad j \in \mathbb{N}_0. \tag{8}$$

This formula could be thought of as a local version of the Grothendieck-Riemann-Roch theorem applied to the forgetful map $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ and a certain sheaf on $\overline{\mathcal{M}}_{g,n+1}$ depending on ℓ (when $\ell = 0$, it is the sheaf of sections of the trivial line bundle). When $\ell = 0$ or $\ell = 1$, our formula agrees modulo boundary terms with the one obtained by Bini [6] using the Grothendieck-Riemann-Roch theorem.

Our results should be compared with the result of Takhtajan and Zograf [30] and Wolpert [35], who gave the two form part of (7) by interpreting it as the first Chern form of the corresponding determinant line bundle equipped with the Quillen connection. As in the compact case, the definition of the Quillen connection makes use of the determinant of the Laplacian. However the presence of cusps induces continuous spectrum for the Laplacian and the usual definition of its determinant via zeta-regularization is necessarily more delicate. Takhtajan and Zograf sidestepped this issue by defining the determinant in terms of the Selberg zeta function, in analogy with the compact case [13, 29]. The precise description of the heat kernel in [31] allows us to proceed along the lines of [22, 28, 12] and extend the zeta function definition to this context via renormalization. Unlike previous efforts (see, e.g., [14], [15] and [26]) this definition does not make use of the hyperbolic structure of the underlying manifold and works more generally for the metrics considered in [31, 2]. Furthermore we show that, for hyperbolic metrics on surfaces with cusps, the resulting zeta-regularized determinant coincides with that defined using the Selberg zeta function up to a universal constant (see theorem 2 and corollary 7.5)

$$\det{}'(\Delta_\ell) = \begin{cases} \alpha_{\ell,g,n} Z_\Sigma(\ell), & \ell \geq 2; \\ \alpha_{\ell,g,n} Z'_\Sigma(1), & \ell = 0, 1; \end{cases} \tag{9}$$

when $\ell \geq 0$, where $\alpha_{\ell,g,n}$ is a constant only depending on ℓ, g and n. With this determinant and thanks to the fact $\ker \overline{\partial}_\ell$ is a holomorphic vector bundle on $T_{g,n}$, the construction of the Quillen connection and the computation of the curvature are essentially as in [5], [8] with only minor changes. In this way, we recover the index formula of [30] (see also [32] and [35]),

$$\frac{\sqrt{-1}}{2\pi}(\nabla^{Q_\ell})^2 = \frac{1}{2\pi i}\left(\int_{\mathcal{T}_{g,n}/T_{g,n}} \mathrm{Ch}\left(T^{-\ell}(\mathcal{T}_{g,n}/T_{g,n})\right) \cdot \mathrm{Td}\left(T\left(\mathcal{T}_{g,n}/T_{g,n}\right)\right)\right)_{[2]} - \sum_{i=1}^n \frac{e_i}{12},$$

see theorem 3 and corollary 8.5 below.

Our approach and the one of Takhtajan and Zograf [30] use substantially the fact that the dimension of the kernel of the family $\overline{\partial}_\ell$ does not jump, so that these kernels fit together into a vector bundle on the Teichmüller space. More generally, one can ask if the work of Bismut, Gillet and Soulé [9, 10, 11] for the determinant of $\overline{\partial}$-operators arising on (compact) Kähler fibrations could be adapted to non-compact situations in order to deal with examples where the rank of the kernel jumps.

The paper is organized as follows. In § 1, we review the definition and main properties of hyperbolic cusp operators. In § 2, we explain the passage from a punctured Riemann surface to a compact Riemann surface with boundary. In § 3, we describe how the $\overline{\partial}$-operator on a punctured Riemann surface can be seen as a Dirac type hyperbolic cusp operator. We also check that Vaillant's formula (5) agrees with the Riemann-Roch theorem in this case. In § 4 and § 5, we make a quick review of Teichmüller theory from our perspective. We then obtain our main result in § 6 by computing the eta forms appearing in the family index formula of [2]. We also compare our formula with the Grothendieck-Riemann-Roch theorem. In § 7, we study the determinant of various Laplacians on Riemann surfaces of finite area and relate them to Selberg's zeta function following [12]. Finally, in § 8, we adapt the standard computation of the curvature of the Quillen connection to our context and compare our result with those of Takhtajan-Zograf [30], Weng [32] and Wolpert [35].

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1. Hyperbolic cusp operators

Let X be a smooth compact manifold with boundary $\partial X \neq \emptyset$. Let $x \in \mathcal{C}^\infty(X)$ be a boundary defining function, that is, x is a positive function in the interior vanishing on the boundary such that its differential dx is nowhere zero on ∂X. For $\epsilon > 0$ sufficiently small, there is an induced collar neighborhood of ∂X in X,

$$c : \partial X \times [0, \epsilon)_x \to N_\epsilon := \{p \in X \quad | \quad x(p) < \epsilon\} \subset X. \tag{1.1}$$

Consider a Riemannian metric g_{hc} in the interior $X \setminus \partial X$ taking the form

$$c^* g_{\mathrm{hc}} = \frac{dx^2}{x^2} + x^2 \pi_L^* g_{\partial X} \tag{1.2}$$

in the collar neighborhood (1.1), where $g_{\partial X}$ is a Riemannian metric on ∂X and $\pi_L : \partial X \times [0, \epsilon)_x \to \partial X$ is the projection on the left factor. Such a metric is called a **product hyperbolic cusp metric** (or product d-metric in the terminology of Vaillant [31]). This is a complete metric on the interior of X, hence the boundary ∂X is at infinity. Notice however that the volume of X is finite with respect to the metric g_{hc}. Following the philosophy of

Melrose, one can get operators that are adapted to this geometry at infinity by considering the space of **hyperbolic cusp vector fields** $\mathcal{V}_{\mathrm{hc}}(X)$, that is, the space of smooth vector fields on X with length uniformly bounded with respect to the metric g_{hc},

$$\mathcal{V}_{\mathrm{hc}}(X) := \{\xi \in \mathcal{C}^\infty(X;TX) \quad | \quad \exists\, c>0 \text{ such that } g_{\mathrm{hc}}(\xi(p),\xi(p)) < c\ \forall\, p \in X\setminus \partial X\}. \tag{1.3}$$

If $z=(z_1,\ldots,z_{n-1})$ are local coordinates on ∂X, then in the collar neighborhood (1.1), a hyperbolic cusp vector field ξ takes the form

$$\xi = ax\frac{\partial}{\partial x} + \sum_{i=1}^{n-1} \frac{b_i}{x}\frac{\partial}{\partial z_i} \tag{1.4}$$

where $a, b_1, \ldots, b_{n-1}$ are smooth functions on X. It is possible to define a vector bundle ${}^{\mathrm{hc}}TX$ on X in such a way that its space of smooth sections is canonically identified with hyperbolic cusp vector fields,

$$\mathcal{C}^\infty(X;{}^{\mathrm{hc}}TX) = \mathcal{V}_{\mathrm{hc}}(X). \tag{1.5}$$

In the interior $X\setminus\partial X$, the vector bundle ${}^{\mathrm{hc}}TX$ is isomorphic to the tangent bundle TX. This identification does not extend to an isomorphism on the boundary of X. The metric g_{hc} naturally induces a metric on ${}^{\mathrm{hc}}TX$ which is also well-defined on the boundary.

A quick check indicates that $\mathcal{V}_{\mathrm{hc}}(X)$ is not closed under the Lie bracket. To define higher order hyperbolic cusp operators, it is convenient to consider the conformally related metric

$$g_{\mathrm{cu}} := \frac{1}{x^2} g_{\mathrm{hc}}. \tag{1.6}$$

The metric g_{cu} is called a **product cusp metric**. One can consider the corresponding **cusp vector fields**

$$\mathcal{V}_{\mathrm{cu}}(X) := x\mathcal{V}_{\mathrm{hc}}(X) = \{\xi \in \mathcal{C}^\infty(X;TX) \quad | \quad \exists\, c>0 \text{ such that } g_{\mathrm{cu}}(\xi(p),\xi(p)) < c\ \forall\, p \in X\setminus \partial X\}. \tag{1.7}$$

Alternatively, one can define cusp vector fields by

$$\mathcal{V}_{\mathrm{cu}}(X) := \{\xi \in \mathcal{C}^\infty(X;TX) \quad | \quad \xi x \in x^2\mathcal{C}^\infty(X)\}, \tag{1.8}$$

which makes it clear that the definition only depends on the choice of boundary defining function x and not on the choice of metric g_{cu}. There is also an associated vector bundle ${}^{\mathrm{cu}}TX$ over X whose space of smooth sections is canonically identified with the space of cusp vector fields,

$$\mathcal{C}^\infty(X;{}^{\mathrm{cu}}TX) = \mathcal{V}_{\mathrm{cu}}(X). \tag{1.9}$$

In the collar neighborhood (1.1), a cusp vector field ξ has to be of the form

$$\xi = ax^2\frac{\partial}{\partial x} + \sum_{i=1}^{n-1} b_i\frac{\partial}{\partial z_i} \tag{1.10}$$

with $a, b_1, \ldots, b_{n-1} \in \mathcal{C}^\infty(X)$. As opposed to $\mathcal{V}_{\mathrm{hc}}(X)$, the space $\mathcal{V}_{\mathrm{cu}}(X)$ is closed under the Lie bracket, so that it is naturally a Lie algebra. Its corresponding universal enveloping algebra is the space $\mathrm{Diff}^*_{\mathrm{cu}}(X)$ of **cusp differential operators**. In the collar neighborhood (1.1), a cusp differential operator of order k, $P \in \mathrm{Diff}^k_{\mathrm{cu}}(X)$, takes the form

$$P = \sum_{l+|\alpha|\le k} p_{l,\alpha} \left(x^2 \frac{\partial}{\partial x} \right)^l \left(\frac{\partial}{\partial z} \right)^\alpha, \quad p_{l,\alpha} \in \mathcal{C}^\infty(X). \tag{1.11}$$

More generally, Mazzeo and Melrose in [21] defined the space of **cusp pseudodifferential operators of order k**, $\Psi^k_{\mathrm{cu}}(X)$. These operators are closed under composition,

$$\Psi^k_{\mathrm{cu}}(X) \circ \Psi^l_{\mathrm{cu}}(X) \subset \Psi^{k+l}_{\mathrm{cu}}(X). \tag{1.12}$$

There is a corresponding **cusp Sobolev space of order** $m \in \mathbb{N}_0$,

$$H^m_{\mathrm{cu}}(X) := \{ f \in L^2_{g_{\mathrm{cu}}}(X) \,|\, Pf \in L^2_{g_{\mathrm{cu}}}(X) \; \forall \; P \in \Psi^m_{\mathrm{cu}}(X) \}. \tag{1.13}$$

One can also consider its weighted version $x^k H^m_{\mathrm{cu}}(X)$ by some power x^k of the boundary defining function. A cusp pseudodifferential operator $P \in \Psi^m_{\mathrm{cu}}(X)$ then defines a bounded linear map

$$P : x^k H^l_{\mathrm{cu}}(X) \to x^k H^{l-m}_{\mathrm{cu}}(X). \tag{1.14}$$

One interesting feature of the cusp operators is that if a cusp pseudodifferential $P \in \Psi^m(X)$ is invertible as a bounded linear map (1.14), then its inverse is given by a cusp operator of order $-m$.

Generalizing the relation $\mathcal{V}_{\mathrm{hc}}(X) = \frac{1}{x}\mathcal{V}_{\mathrm{cu}}(X)$, one can define the space of **hyperbolic cusp pseudodifferential operators of order** m by

$$\Psi^m_{\mathrm{hc}}(X) := x^{-m} \Psi^m_{\mathrm{cu}}(X). \tag{1.15}$$

A hyperbolic cusp operator $P \in \Psi^m_{\mathrm{hc}}(X)$ naturally induces a bounded linear map

$$P : x^k H^l_{\mathrm{cu}}(X) \to x^{k-m} H^{l-m}_{\mathrm{cu}}(X). \tag{1.16}$$

So far we have considered operators acting on functions on X, but if $E \to X$ and $F \to X$ are complex vector bundles on X, it is no more difficult to define the space of hyperbolic cusp operators $\Psi^*_{\mathrm{hc}}(X; E, F)$ acting from sections of E to sections of F.

In [21], Mazzeo and Melrose gave a very elegant criterion to determine when a cusp operator is Fredholm. They first introduced a notion of principal symbol adapted to the geometry at infinity, that is, involving the cosphere bundle $S^*({}^{\mathrm{cu}}TX)$ of ${}^{\mathrm{cu}}TX$,

$$\sigma_k : \Psi^k_{\mathrm{cu}}(X; E, F) \to \mathcal{C}^\infty(S^*({}^{\mathrm{cu}}TX); \hom(\pi^* E, \pi^* F)) \tag{1.17}$$

where $\pi : S^{*\mathrm{cu}}TX \to X$ is the bundle projection. A cusp operator $A \in \Psi^k_{\mathrm{cu}}(X; E, F)$ is said to be **elliptic** if its principal symbol $\sigma_k(A)$ is invertible.

In that case, by a standard construction, one can obtain a parametrix $B \in \Psi^k_{\mathrm{cu}}(X;F,E)$ such that

$$BA - \mathrm{Id}_E \in \Psi^{-\infty}_{\mathrm{cu}}(X;E), \quad AB - \mathrm{Id}_F \in \Psi^{-\infty}_{\mathrm{cu}}(X;F). \tag{1.18}$$

However, since elements of $\Psi^{-\infty}_{\mathrm{cu}}(X;E)$ are not compact in general, this does not insure that the operator A is Fredholm. One needs some extra decay at infinity for the error term to be compact. Precisely, the subset of compact operators in $\Psi^{-\infty}(X;E)$ is given by $x\Psi^{-\infty}(X;E)$. It is possible to insure the error term is in that subset provided A is 'invertible at infinity'. This condition is determined by the normal operator map

$$N : \Psi^k_{\mathrm{cu}}(X;E,F) \to \Psi^k_{\mathrm{sus}}(\partial X;E,F) \tag{1.19}$$

where $\Psi^k_{\mathrm{sus}}(\partial X;E,F)$ is the space of suspended operators of order k introduced by Melrose in [23]. These are operators on $\partial X \times \mathbb{R}$ which are translation invariant in the $\mathbb{R}$ direction. Essentially, the normal operator $N(A)$ of A is its asymptotically translation invariant part at infinity. The criterion of Mazzeo and Melrose can now be stated as follows.

Proposition 1.1 (Mazzeo-Melrose). *A cusp operator $A \in \Psi^k_{\mathrm{cu}}(X;E,F)$ is Fredholm if and only if it is elliptic and its normal operator $N(A)$ is invertible.*

For hyperbolic cusp operators, the situation is much more delicate. For simplicity, let us restrict to a first order hyperbolic cusp **differential** operator $\eth_{\mathrm{hc}} \in \Psi^1_{\mathrm{hc}}(X;E,F)$. Then $x\eth_{\mathrm{hc}} \in \Psi^1_{\mathrm{cu}}(X;E,F)$ is a cusp operator and we can use proposition 1.1 to determine whether or not $x\eth_{\mathrm{hc}}$ is Fredholm. If it is Fredholm, then it is not hard to see that $\eth_{\mathrm{hc}}$ is Fredholm as well. In fact, in that case, the spectrum of $\eth_{\mathrm{hc}}$ is then necessarily discrete since its parametrix in $x\Psi^{-1}_{\mathrm{cu}}(X;F,E)$ is a compact operator.

However, even if $x\eth_{\mathrm{hc}}$ is not Fredholm, it is still possible for $\eth_{\mathrm{hc}}$ to be Fredholm. Define the **vertical family** of $\eth_{\mathrm{hc}}$ to be

$$\eth^V_{\mathrm{hc}} := (x\eth_{\mathrm{hc}})|_{\partial X} \in \Psi^1(\partial X;E,F). \tag{1.20}$$

When $\eth_{\mathrm{hc}}$ is a self-adjoint Dirac type operator with $E = F$ a Clifford bundle, the vertical family $\eth^V_{\mathrm{hc}}$ is invertible if and only if the normal operator $N(x\eth_{\mathrm{hc}})$ is invertible. In his thesis [31], Vaillant gave the following criterion to determine if a Dirac-type self-adjoint operator $\eth_{\mathrm{hc}}$ is Fredholm. The vertical family does not have to be invertible, but if it is not, Vaillant defined another operator $\eth^H_{\mathrm{hc}}$ acting on the finite dimensional vector space $\mathcal{K} := \ker \eth^V_{\mathrm{hc}}$ and called the **horizontal family**. If Π_0 denotes the projection from $L^2(\partial X;E)$ onto $\mathcal{K}$, then the horizontal family is defined by extending an element $\xi \in \mathcal{K}$ into the interior to an element $\widetilde{\xi} \in \mathcal{C}^\infty(X;E)$ and then applying $\eth_{\mathrm{hc}}$ and Π_0,

$$\eth^H_{\mathrm{hc}}\xi := \Pi_0 \left(\eth_{\mathrm{hc}}\widetilde{\xi}\Big|_{\partial X} \right). \tag{1.21}$$

In his thesis [31], Vaillant gave the following criterion.

Proposition 1.2 (Vaillant [31], §3). *A Dirac type self-adjoint operator $\eth_{\mathrm{hc}} \in \Psi^1_{\mathrm{hc}}(X;E)$ is Fredholm if and only if $\eth^H_{\mathrm{hc}}$ is invertible. Moreover, the continuous spectrum of $\eth_{\mathrm{hc}}$ is governed by $\eth^H_{\mathrm{hc}}$ with bands of continuous spectrum starting at the eigenvalues of $\eth^H_{\mathrm{hc}}$ and going to infinity.*

2. The boundary compactification of a Riemann surface with puncture

Let Σ be a Riemann surface of type (g,n), that is, $\Sigma = \overline{\Sigma} \setminus \{x_1,\dots,x_n\}$ where $\overline{\Sigma}$ is a compact Riemann surface of genus g and $x_1,\dots,x_n$ are pairwise distinct points on $\overline{\Sigma}$. We will assume that $2g+n \geq 3$. The surface $\overline{\Sigma}$ is a compactification of Σ. An alternative way of compactifying the Riemann surface Σ is to consider the radial blow up Σ_b of $\overline{\Sigma}$ at the points $\{x_1,\dots,x_n\}$ with blow-down map

$$\beta : \Sigma_b \to \overline{\Sigma}. \tag{2.1}$$

This gives a compactification of Σ in which each puncture is replaced by a circular boundary. The Riemann surface with boundary Σ_b also comes equipped with a natural choice of boundary defining function $\rho \in \mathcal{C}^\infty(\Sigma_b)$ as we will see. This choice is dictated by the uniformization theorem for Riemann surfaces.

Recall that, by the uniformization theorem, there is a canonical hyperbolic metric g_Σ on Σ obtained by taking the unique metric of constant scalar curvature equal to -1 in the conformal class defined by the complex structure of Σ. Consider the upper-half plane

$$\mathbb{H} = \{x+iy \in \mathbb{C} \mid y>0\} \tag{2.2}$$

equipped with the Poincaré metric

$$g_{\mathbb{H}} := \frac{dx^2+dy^2}{y^2}. \tag{2.3}$$

Let Γ_∞ be the discrete Abelian group generated by the parabolic isometry $z \mapsto z+1$. The *horn* is the quotient

$$H := \Gamma_\infty \setminus \mathbb{H}. \tag{2.4}$$

Via the change of variable $r = \frac{1}{y}$, one sees that the horn is isometric to $(0,+\infty)_r \times \mathbb{R}/\mathbb{Z}$ equipped with the metric

$$\frac{dr}{r^2} + r^2 dx^2. \tag{2.5}$$

A *cusp end* is a subspace of H of the form $(0,a] \times \mathbb{R}/\mathbb{Z}$. Near a puncture x_i of Σ, the geometry of (Σ, g_Σ) is modeled on a cusp end. That is, around each puncture x_i, there exists a neighborhood $N_i \subset \Sigma$ and an isometry

$$\varphi_i : N_i \to C_i \tag{2.6}$$

with a cusp end $C_i = (0, \frac{1}{y_i}] \times \mathbb{R}/\mathbb{Z}$. Each cusp end has a natural compactification

$$\overline{C}_i = \left[0, \frac{1}{y_i}\right]_{r_i} \times \mathbb{R}/\mathbb{Z} \tag{2.7}$$

where the coordinate r_i can be seen as a boundary defining function for the boundary $\{0\} \times \mathbb{R}/\mathbb{Z} \subset \overline{C}_i$. This boundary defining function can in fact be defined intrinsically in terms of the hyperbolic metric (2.5). Indeed, we define a *horocycle* to be an embedded circle in a cusp end which is perpendicular to all geodesics emanating from the cusp. This definition is formulated purely in terms of the metric. On the other hand, as one can check, the horocycles are precisely given by the level sets of the function r_i. Moreover, the value of the function r_i on a horocycle $\gamma = \{u\} \times \mathbb{R}/\mathbb{Z}$ is also determined by the hyperbolic metric. It is the area of the smaller cusp end $(0, u) \times \mathbb{R}/\mathbb{Z}$, namely

$$r_i(u, v) = \operatorname{area}((0, u) \times \mathbb{R}/\mathbb{Z})) = \int_0^u \int_{\mathbb{R}/\mathbb{Z}} dr dx = u. \tag{2.8}$$

Thus, intuitively, the boundary defining function r_i is the 'area function' for the cusp end C_i. The compactification $\overline{C}_i$ induces a corresponding compactification $\overline{N}_i$ via the isometry (2.6), and thus a compactification Σ_{hc} of Σ into a compact surface with boundary naturally diffeomorphic to Σ_b. To get a global boundary defining function, choose a smooth non-decreasing function $\chi \in \mathcal{C}^\infty([0, +\infty))$ such that

$$\chi(x) := \begin{cases} x, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ 1, & \text{if } x \geq 1, \end{cases} \tag{2.9}$$

and consider $\chi_\epsilon(x) := \epsilon\chi(\frac{x}{\epsilon})$ for $0 < \epsilon < \min\{\frac{1}{y_1}, \ldots, \frac{1}{y_n}\}$. On each (compactified) cusp end $\overline{C}_i$, consider the function $\chi_\epsilon(r_i)$. Then the function

$$\rho_{\Sigma,\epsilon}(\sigma) := \begin{cases} \varphi_i^*(\chi_\epsilon \circ r_i)(\sigma), & \text{if } \sigma \in \overline{N}_i, \quad i \in \{1, \ldots, n\}; \\ \epsilon, & \text{otherwise}; \end{cases} \tag{2.10}$$

is a boundary defining function for $\partial\Sigma_{\mathrm{hc}}$ in Σ_{hc}. Since the choice of the number ϵ is not of primary importance, we will usually denote the function $\rho_{\Sigma,\epsilon}$ simply by ρ_Σ. With respect to this boundary defining function, the hyperbolic metric g_Σ is a product hyperbolic metric. That is, in the coordinates (x, ρ_Σ) on N_i, it is of the form

$$g_\Sigma = \frac{d\rho_\Sigma^2}{\rho_\Sigma^2} + \rho_\Sigma^2 dx^2 \tag{2.11}$$

near the boundary.

3. The $\overline{\partial}$-operator as a Dirac-type hc-operator

Let $K := \Lambda^{1,0}_{\Sigma}$ denote the canonical line bundle on Σ. This line bundle and all of its tensor powers K^{ℓ} have natural holomorphic structures. In particular, for each $\ell \in \mathbb{Z}$, there is a well-defined $\overline{\partial}$ operator

$$\overline{\partial}_{\ell} : \mathcal{C}^{\infty}(\Sigma; K^{\ell}) \to \mathcal{C}^{\infty}(\Sigma; \Lambda^{0,1}_{\Sigma} \otimes K^{\ell}), \tag{3.1}$$

where $\Lambda^{0,1}_{\Sigma} \to \Sigma$ is the bundle of $(0,1)$-forms on Σ. In a cusp end C_i where the canonical line bundle is trivialized by the holomorphic section dz, it takes the form

$$\begin{aligned} d\overline{z}\frac{\partial}{\partial \overline{z}} &= (dx - idy)\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ &= (dx + i\frac{dr}{r^2})\frac{1}{2}\left(\frac{\partial}{\partial x} - ir^2\frac{\partial}{\partial r}\right), \quad r = \frac{1}{y}, \\ &= \frac{1}{2}\left(rdx + i\frac{dr}{r}\right)\left(\frac{1}{r}\frac{\partial}{\partial x} - ir\frac{\partial}{\partial r}\right). \end{aligned} \tag{3.2}$$

Thus, near the boundary $\partial\Sigma_{\mathrm{hc}}$, the $\overline{\partial}$-operator is of the form

$$\overline{\partial} = \frac{1}{2}\left(\rho_{\Sigma}dx + i\frac{d\rho_{\Sigma}}{\rho_{\Sigma}}\right)\left(\frac{1}{\rho_{\Sigma}}\frac{\partial}{\partial x} - i\rho_{\Sigma}\frac{\partial}{\partial \rho_{\Sigma}}\right). \tag{3.3}$$

Since $\frac{1}{\rho_{\Sigma}}\frac{\partial}{\partial x} - i\rho_{\Sigma}\frac{\partial}{\partial \rho_{\Sigma}}$ is a hc-operator and $\frac{1}{2}(\rho_{\Sigma}dx + i\frac{d\rho_{\Sigma}}{\rho_{\Sigma}})$ is naturally a section of ${}^{\mathrm{hc}}T^*\Sigma \otimes_{\mathbb{R}} \mathbb{C}$, we see that the $\overline{\partial}_{\ell}$-operator naturally extends to give a hc-operator

$$\overline{\partial}_{\ell} : \mathcal{C}^{\infty}(\Sigma_{\mathrm{hc}}; {}^{\mathrm{hc}}K^{\ell}) \to \frac{1}{\rho_{\Sigma}}\mathcal{C}^{\infty}(\Sigma_{hc}; {}^{\mathrm{hc}}\Lambda^{0,1}_{\Sigma} \otimes {}^{\mathrm{hc}}K^{\ell}) \tag{3.4}$$

where ${}^{\mathrm{hc}}\Lambda^{0,1}_{\Sigma}$ is the complex conjugate of ${}^{\mathrm{hc}}K$ and ${}^{\mathrm{hc}}K \subset {}^{\mathrm{hc}}T^*\Sigma \otimes_{\mathbb{R}} \mathbb{C}$ is such that it is identified with K in the interior of Σ_{hc} and it is trivialized by the section $\rho_{\Sigma}dz = \rho_{\Sigma}dx - i\frac{d\rho_{\Sigma}}{\rho_{\Sigma}}$ near each connected component of the boundary. The metric g_{Σ} induces a Hermitian metric on K^{ℓ} and $\Lambda^{0,1}_{\Sigma}$, as well as on ${}^{\mathrm{hc}}K^{\ell}$ and ${}^{\mathrm{hc}}\Lambda^{0,1}_{\Sigma}$. We denote by $\mathcal{H}_{\ell,i}$ the Hilbert space of square integrable sections of ${}^{\mathrm{hc}}K^{\ell} \otimes ({}^{\mathrm{hc}}\Lambda^{0,1})^{i}$ with respect to the natural scalar product

$$\langle f_1, f_2\rangle_{\mathcal{H}_{\ell,i}} := \int_{\Sigma_{\mathrm{hc}}} \langle f_1(\sigma), f_2(\sigma)\rangle_{g_{\Sigma}} dg_{\Sigma}(\sigma) \tag{3.5}$$

where dg_{Σ} is the natural extension of the volume form of g_{Σ} on Σ_{hc}.

The operator $\overline{\partial}_{\ell}$ is Fredholm. To see this, recall (see for instance proposition 3.67 in [5]) that

$$D_{\ell} := \sqrt{2}(\overline{\partial}_{\ell} + \overline{\partial}_{\ell}^{*}) \tag{3.6}$$

is a Dirac type operator induced by the Chern connection on K^{ℓ} with Clifford action on $\nu \in \mathcal{C}^{\infty}(\Sigma_{\mathrm{hc}}; {}^{\mathrm{hc}}\Lambda_{\Sigma})$ given by

$$c(f)\nu = \sqrt{2}(\varepsilon(f^{0,1}) - \iota(f^{1,0}))\nu, \quad f \in \mathcal{C}^{\infty}(\Sigma_{\mathrm{hc}}; {}^{\mathrm{hc}}\Lambda_{\Sigma}), \tag{3.7}$$

where $\varepsilon(f^{0,1})$ denotes exterior multiplication by $f^{0,1}$. The operator D_ℓ is formally self-adjoint. The vertical family D_ℓ^V of D_ℓ is given by

$$c(du)\frac{\partial}{\partial u} \tag{3.8}$$

acting on $\mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}; {}^{\mathrm{hc}}K^\ell \oplus {}^{\mathrm{hc}}\Lambda^{0,1}_\Sigma \otimes {}^{\mathrm{hc}}K^\ell)$ on each circular boundary component of Σ_{hc}, where $u = -x$ is such that $\{\frac{\partial}{\partial u}\big|_\sigma\}$ is an oriented orthonormal basis of $T_\sigma \partial\Sigma_{\mathrm{hc}}$ for each $\sigma \in \partial\Sigma_{\mathrm{hc}}$. In particular, $\mathcal{K} = \ker(D_\ell^V)$ is a complex vector space of dimension $2n$. By proposition 1.2, we need to show that the horizontal family $D_\ell^H : \mathcal{K} \to \mathcal{K}$ is invertible.

Proposition 3.1. *On each circular boundary component of* Σ_{hc}*, the horizontal family is given by*

$$D_\ell^H = \left(\ell - \frac{1}{2}\right) ic(du).$$

Proof. The bundle on which D_ℓ acts is

$$\left(\underline{\mathbb{C}} \oplus \Lambda^{0,1}\Sigma\right) \otimes K^\ell.$$

Choose a spin structure on Σ and let S be the corresponding spinor bundle. It is well-known (see for instance [19]) that, seen as complex line bundle, S is a square root of the canonical line bundle K so that

$$S \otimes_{\mathbb{C}} S = K.$$

Moreover, we have also that

$$\left(\underline{\mathbb{C}} \oplus \Lambda^{0,1}\Sigma\right) \cong S \otimes_{\mathbb{R}} S^*.$$

Thus the operator D_ℓ acts on

$$S \otimes_{\mathbb{R}} \left(S^* \otimes_{\mathbb{C}} K^\ell\right),$$

which means that D_ℓ is a Dirac operator twisted by the bundle $S^* \otimes_{\mathbb{C}} K^\ell$. As a bundle with connection, the bundle $S^* \otimes K^\ell$ certainly does not have a product structure near the boundary since it has non-zero curvature. Thus, according to Proposition 3.15, p.44 in [31], the horizontal family D_ℓ^H at each cusp is given by

$$-iRc\left(\frac{\partial}{\partial u}\right) = -i\left(\frac{1}{2} - \ell\right) c\left(\frac{\partial}{\partial u}\right)$$

where $iRdg_\Sigma$ is the curvature of the complex vector bundle $S^* \otimes_{\mathbb{C}} K^\ell$ (cf. (3.15)). Collecting the contributions at each cusp end, we get the desired result. □

This gives the following corollary.

Corollary 3.2. *The operators*

$$D_\ell = \sqrt{2}\begin{pmatrix} 0 & \overline{\partial}_\ell^* \\ \overline{\partial}_\ell & 0 \end{pmatrix}, \quad \overline{\partial}_\ell, \text{ and } \overline{\partial}_\ell^*$$

are Fredholm.

Notice that proposition 3.1 is also consistent with the well-known fact that the band of continuous spectrum of the Hodge Laplacian D_ℓ^2 starts at $\left(\frac{1}{2}-\ell\right)^2$ and goes to infinity.

In his thesis [31], Vaillant obtained a general formula for the index of a Dirac type operator on a fibred hyperbolic cusp operator. For the index of the operator $\overline{\partial}_\ell$, this formula is given by the usual Atiyah-Singer integrand together with two corrections coming from the boundary, namely the eta invariants associated to the vertical family of $\overline{\partial}_\ell$ and the horizontal family D_ℓ^H,

$$\operatorname{ind}(\overline{\partial}_\ell) = \int_{\Sigma_{\mathrm{hc}}} \mathrm{Ch}({}^{\mathrm{hc}}K^\ell)\,\mathrm{Td}({}^{\mathrm{hc}}K^{-1}) - \frac{1}{2}\eta(D_\ell^V) - \frac{1}{2}\eta(D_\ell^H). \tag{3.9}$$

The eta invariant of the vertical family is easily seen to be zero. This is because modulo standard identifications, $\eta(D_\ell^V)$ corresponds to n times the eta invariant of the self-adjoint operator

$$\frac{1}{i}\frac{\partial}{\partial x} = i\frac{\partial}{\partial u} : \mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}) \to \mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}). \tag{3.10}$$

But the spectrum of $\frac{1}{i}\frac{\partial}{\partial x}$ is $2\pi\mathbb{Z}$ and its eta functional

$$\eta(\frac{1}{i}\frac{\partial}{\partial x}, s) = \sum_{k\neq 0} 2\pi k |2\pi k|^{-s}, \quad \operatorname{Re} s >> 0 \tag{3.11}$$

is identically zero. Thus its spectral asymmetry or eta invariant, which is the value at $s=0$ of the analytic continuation of $\eta(i\frac{\partial}{\partial x}, s)$, is zero. The corresponding eta invariant $\eta(D_\ell^V) = n\eta(i\frac{\partial}{\partial x})$ therefore vanishes. For the computation of the spectral asymmetry of D_ℓ^H, there is no regularization involved since D_ℓ^H is just an endomorphism of a finite dimensional vector space. From proposition 3.1, we compute directly (see [2, (4.14)]) that

$$\eta(D_\ell^H) = n \operatorname{sign}\left(\ell - \frac{1}{2}\right). \tag{3.12}$$

The index is therefore given by

$$\operatorname{ind}(\overline{\partial}_\ell) = \int_{{}^{\mathrm{hc}}\Sigma} \mathrm{Ch}({}^{\mathrm{hc}}K^\ell)\,\mathrm{Td}({}^{\mathrm{hc}}K^{-1}) + \frac{n}{2}\operatorname{sign}\left(\frac{1}{2} - \ell\right). \tag{3.13}$$

The integral is also easy to compute. Let Θ_Σ denote the curvature of ${}^{\mathrm{hc}}T^{1,0}\Sigma$. Then the integrand is given by

$$
\begin{aligned}
\mathrm{Ch}({}^{\mathrm{hc}}K^\ell)\,\mathrm{Td}({}^{\mathrm{hc}}T^{1,0}\Sigma) &= \left(e^{-\frac{\ell i}{2\pi}\Theta_\Sigma}\right)\left(\frac{\frac{i}{2\pi}\Theta_\Sigma}{1-e^{-\frac{i}{2\pi}\Theta_\Sigma}}\right) \\
&= 1+\left(\frac{1}{2}-\ell\right)\frac{i}{2\pi}\Theta_\Sigma.
\end{aligned} \tag{3.14}
$$

By a standard computation (see for instance p.77 in [16]), we know that

$$
\frac{i}{2\pi}\Theta_\Sigma = \frac{\kappa}{2\pi}dg_\Sigma = -\frac{1}{2\pi}dg_\Sigma \tag{3.15}
$$

where $\kappa=-1$ is the Gaussian curvature of g_Σ. By the Gauss-Bonnet theorem applied to Σ, we get that

$$
\begin{aligned}
\mathrm{ind}(\overline{\partial}_\ell) &= \left(\frac{1}{2}-\ell\right)\int_{\Sigma_{\mathrm{hc}}}\frac{\kappa}{2\pi}dg_\Sigma + \frac{n}{2}\,\mathrm{sign}\left(\frac{1}{2}-\ell\right) \\
&= \left(\frac{1}{2}-\ell\right)\chi(\Sigma) + \frac{n}{2}\,\mathrm{sign}\left(\frac{1}{2}-\ell\right) \\
&= \left(\frac{1}{2}-\ell\right)(2-2g-n) + \frac{n}{2}\,\mathrm{sign}\left(\frac{1}{2}-\ell\right).
\end{aligned} \tag{3.16}
$$

This gives the following formula.

Proposition 3.3. *The index of $\overline{\partial}_\ell$ is given by*

$$
\mathrm{ind}(\overline{\partial}_\ell) = \begin{cases} (2\ell-1)(g-1)+\ell n, & \ell\le 0, \\ (2\ell-1)(g-1)+(\ell-1)n, & \ell>0. \end{cases}
$$

In fact, using the Riemann-Roch theorem on the compact Riemann surface $\overline{\Sigma}$, it is also possible to compute explicitly the dimension of the kernel and the cokernel of $\overline{\partial}_\ell$ (cf. p.404 in [30]). By definition, an element of $f\in\ker\overline{\partial}_\ell$ is a holomorphic section of K^ℓ, so in each cusp end N_j, it has a Laurent series expansion

$$
f(z) = \sum_{k=-\infty}^{\infty} a_k^{(j)} e^{2\pi i k z}(dz)^\ell. \tag{3.17}
$$

When $\ell>0$, this expansion has to be of the form

$$
f(z) = \sum_{k=1}^{\infty} a_k^{(j)} e^{2\pi i k z}(dz)^\ell \tag{3.18}
$$

in order for f to be an element of $\mathcal{H}_{\ell,0}$. Such an f is said to be a **cusp form** of weight $(2\ell,0)$. When $\ell\le 0$, we can also have a constant coefficient in the series,

$$
f(z) = \sum_{k=0}^{\infty} a_k^{(j)} e^{2\pi i k z}(dz)^\ell. \tag{3.19}
$$

When $\ell = 0$, using the coordinate $\zeta := e^{2\pi i z}$ near each puncture x_j in $\overline{\Sigma}$, we see that such a f naturally extends to give a holomorphic function on $\overline{\Sigma}$. It is therefore constant, so that $\dim_{\mathbb{C}} \ker \overline{\partial}_0 = 1$. When $\ell \geq 1$, the section f takes the form

$$f(z) = \sum_{k=1}^{\infty} a_k^{(j)} \zeta^k \left(\frac{d\zeta}{2\pi i \zeta} \right)^{\ell} \tag{3.20}$$

in the coordinate ζ near the puncture x_j. Thus, it naturally extends to a meromorphic section of $\overline{K}^{\ell} \to \overline{\Sigma}$ with poles of order not exceeding $\ell - 1$ at each puncture $x_1, \ldots, x_n$ and holomorphic elsewhere. Conversely, such a meromorphic section corresponds to an element of $\ker \overline{\partial}_{\ell}$. We can thus compute $\dim_{\mathbb{C}} \ker \overline{\partial}_{\ell}$ by applying the Riemann-Roch theorem on $\overline{\Sigma}$ to the line bundle

$$L_D \otimes \overline{K}^{\ell} \tag{3.21}$$

where L_D is the holomorphic line bundle associated to the divisor

$$D = \sum_{i=1}^{n} (\ell - 1) x_i \quad \text{on } \overline{\Sigma}. \tag{3.22}$$

This gives

$$\begin{aligned} \dim_{\mathbb{C}} \ker \overline{\partial}_{\ell} &= h^0(L_D \otimes \overline{K}^{\ell}) \\ &= h^0(\overline{K} \otimes (L_D \otimes \overline{K}^{\ell})^{-1}) + \deg(L_D \otimes \overline{K}^{\ell}) - g + 1 \\ &= h^0(\overline{K} \otimes (L_D \otimes \overline{K}^{\ell})^{-1}) + n(\ell - 1) + (2\ell - 1)(g - 1), \end{aligned} \tag{3.23}$$

where $h^0(L)$ denotes the dimension of the space of holomorphic sections of the holomorphic line bundle L. Now we compute that

$$\deg(\overline{K} \otimes L_D^{-1} \otimes \overline{K}^{-\ell}) = -(\ell - 1)(2g + n - 2). \tag{3.24}$$

When $\ell = 1$, $\overline{K} \otimes (L_D \otimes \overline{K}^{\ell})^{-1}$ is the trivial line bundle, so $h^0(\overline{K} \otimes (L_D \otimes \overline{K}^{\ell})^{-1}) = 1$ in this case. When $\ell > 1$, $\deg(K \otimes L_D^{-1} \otimes K^{-\ell}) < 0$ since we assume that $2g + n \geq 3$, and therefore $h^0(K \otimes L_D^{-1} \otimes K^{-\ell}) = 0$. Finally, when $\ell < 0$, elements of $\ker \overline{\partial}_{\ell}$ correspond to holomorphic sections of $K_{\overline{\Sigma}}^{\ell}$ with zeros of degree at least $-\ell$ at each puncture. These in turn correspond to the holomorphic sections of a holomorphic line bundle of negative degree (since $2g + n \geq 3$), so that $\ker \overline{\partial}_{\ell} = 0$ in that case. Hence, we see that the dimension of the kernel of $\overline{\partial}_{\ell}$ is given by

$$\dim \ker \overline{\partial}_{\ell} = \begin{cases} 0, & \ell < 0, \\ 1, & \ell = 0, \\ g, & \ell = 1 \\ (2\ell - 1)(g - 1) + n(\ell - 1), & l \geq 2. \end{cases} \tag{3.25}$$

Comparing with the index (3.16), we also get that

$$\dim\ker\overline{\partial}_\ell^* = \begin{cases} -(2\ell-1)(g-1)-n\ell, & \ell<0,\\ g, & \ell=0,\\ 1, & \ell=1\\ 0, & l\ge 2. \end{cases} \tag{3.26}$$

These formulas are consistent with Kodaira-Serre duality, which asserts in this case that $\ker\overline{\partial}_\ell^* \cong \ker\overline{\partial}_{1-\ell}$.

4. The Teichmüller space and the Teichmüller universal curve

So far we have assumed that the complex structure on Σ was fixed. By changing the complex structure, one can get instead a family of $\overline{\partial}_\ell$ operators. The universal case is obtained by considering all at once the moduli space of all complex structures on a surface of type (g,n), two complex structures being identified whenever there is a conformal transformation between them homotopic to the identity. It is called the Teichmüller space of Riemann surfaces of genus g with n punctures and is denoted $T_{g,n}$. It is a complex manifold of complex dimension $3g-3+n$ which can be identified with an open set of $\mathbb{C}^{3g-3+n}$. The Teichmüller space $T_{g,n}$ comes together with a universal bundle, the universal Teichmüller curve $\mathcal{T}_{g,n}$ with bundle projection

$$p:\mathcal{T}_{g,n}\to T_{g,n} \tag{4.1}$$

and fibre $p^{-1}([\Sigma])$ the Riemann surface Σ of type (g,n) corresponding to the point $[\Sigma]\in T_{g,n}$. Denote by $T_v^{i,j}\mathcal{T}_{g,n}\to\mathcal{T}_{g,n}$ the vertical (i,j) tangent bundle of the fibration (4.1) for $i,j\in\{0,1\}$. On each fibre $\Sigma := p^{-1}([\Sigma])$, the restriction of $T_v^{i,j}\mathcal{T}_{g,n}$ is canonically identified with $T^{i,j}\Sigma$. Denote by $\Lambda_v^{i,j}\to\mathcal{T}_{g,n}$ the dual of $T_v^{i,j}$. On each fibre we also have a $\overline{\partial}$-operator. These operators fit together to give a family of operators

$$\overline{\partial}_\ell \in \rho^{-1}\Psi^1_{\mathrm{cu}}(\mathcal{T}_{g,n}/T_{g,n}; (\Lambda_v^{1,0})^\ell, \Lambda_v^{0,1}\otimes(\Lambda_v^{1,0})^\ell) \tag{4.2}$$

where ρ is an appropriate boundary defining function (whose precise definition we postpone to (5.16)). Each element of the family is a Fredholm operator so that we have a family index in $K^0(T_{g,n})$,

$$\operatorname{ind}(\overline{\partial}_\ell)\in K^0(T_{g,n}). \tag{4.3}$$

Since the Teichmüller space is contractible, this families index really only encodes the numerical index of any member of the family under the identification $K^0(T_{g,n})\cong K^0(\mathrm{pt})\cong\mathbb{Z}$. Still, it is possible to exhibit an explicit representative of the K-class $\operatorname{ind}(\overline{\partial}_\ell)\in K^0(T_{g,n})$, providing in this way a local description of the family index. This is because, according to (3.25) and (3.26), the dimensions of the kernel and the cokernel of elements of the family $\overline{\partial}_\ell$ are always the same (they only depend on ℓ, g and n, not on the complex structure). This means that

$$\ker\overline{\partial}_\ell\to T_{g,n} \quad\text{and}\quad \ker\overline{\partial}_\ell^*\to T_{g,n} \tag{4.4}$$

form complex vector bundles on $T_{g,n}$ and the family index of $\overline{\partial}_\ell$ can then be expressed as the virtual difference of these two vector bundles,

$$\operatorname{ind}\overline{\partial}_\ell = [\ker\overline{\partial}_\ell] - [\ker\overline{\partial}_\ell^*] \in K^0(T_{g,n}). \tag{4.5}$$

In fact, as we will recall in a moment, these vector bundles both come equipped with a natural connection. We can therefore express their respective Chern characters at the level of forms. This provides a local description of the Chern character of the family index

$$\operatorname{Ch}(\operatorname{ind}\overline{\partial}_\ell) := \operatorname{Ch}(\ker\overline{\partial}_\ell) - \operatorname{Ch}(\ker\overline{\partial}_\ell^*) \in \mathcal{C}^\infty(T_{g,n}, \Lambda^{\mathrm{ev}}(T_{g,n})). \tag{4.6}$$

On the Teichmüller space itself, this local description of the index does not contain more cohomological information than (3.16). However, the local descriptions (4.5) and (4.6) are invariant under the action of the Teichmüller modular group $\operatorname{Mod}_{g,n}$. This means that these local descriptions descend to the moduli space $T_{g,n}/\operatorname{Mod}_{g,n}$ (in the sense of orbifolds), which typically has a non-trivial topology as well as singularities.

5. The canonical connection on the universal Teichmüller curve

The fibration $p : \mathcal{T}_{g,n} \to T_{g,n}$ comes together with a canonical connection $\mathcal{P}$. To describe this connection, one possible approach is to describe Riemann surfaces as certain quotients of the upper half-plane $\mathbb{H}$. If Σ is a Riemann surface of genus g with n punctures, then it can be represented as a quotient $\Gamma\setminus\mathbb{H}$ of the upper half-plane by the action of a torsion-free finitely generated Fuchsian group Γ. The group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is of type (g,n), which is to say it is generated by $2g$ hyperbolic transformations $A_1, B_1, \ldots, A_g, B_g$ and n parabolic transformations $S_1, \ldots, S_n$ satisfying the single relation $A_1B_1A_1^{-1}B_1^{-1}\cdots A_gB_gA_g^{-1}B_g^{-1}S_1\cdots S_n = 1$. Since $\mathbb{H}$ is simply connected, in fact contractible, it is the universal cover of Σ under the quotient map $\mathbb{H} \to \Gamma\setminus\mathbb{H}$. From this perspective, the canonical hyperbolic metric g_Σ associated to the (conformal structure of the) complex structure is precisely the metric on $\Gamma\setminus\mathbb{H}$ induced from the Poincaré metric

$$g_{\mathbb{H}} := \frac{dx^2+dy^2}{y^2} \text{ on } \mathbb{H}. \tag{5.1}$$

The punctures of Σ then correspond to the image of the fixed points $z_1, \cdots, z_n$ in $\mathbb{R}\cup\{\infty\}$ of the parabolic transformations $S_1, \ldots, S_n$ under the quotient map $\mathbb{H} \to \Gamma\setminus\mathbb{H}$. Let Γ_i be the cyclic subgroup of Γ generated by the parabolic transformation S_i for $i = 1, \ldots, n$. It can be identified with the cyclic group Γ_∞ by choosing $\sigma_i \in \mathrm{PSL}(2,\mathbb{R})$ such that $\sigma_i\infty = z_i$, so that

$$\sigma_i^{-1}S_i\sigma_i = \begin{pmatrix} 1 & \pm1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_i^{-1}\Gamma_i\sigma_i = \Gamma_\infty. \tag{5.2}$$

On Σ, sections of $(\Lambda_\Sigma^{1,0})^\ell \otimes ((\Lambda_\Sigma)^{0,1})^m$ correspond to automorphic forms of weight $(2\ell, 2m)$ with respect to the group Γ, that is, functions $f : \mathbb{H} \to \mathbb{C}$

such that

$$f(\gamma z)\gamma'(z)^{\ell}\overline{\gamma'(z)}^{m} = f(z) \quad \forall z \in \mathbb{H},\ \forall \gamma \in \Gamma. \tag{5.3}$$

For instance, the natural Kähler metric associated to the hyperbolic metric g_Σ, seen as a section of $\Lambda^{1,0}_\Sigma \otimes \Lambda^{0,1}_\Sigma$, corresponds to the automorphic form of weight $(2,2)$

$$\frac{1}{y^2} \text{ on } \mathbb{H}. \tag{5.4}$$

In the correspondence between Riemann surfaces and quotients of $\mathbb{H}$, a change of complex structure corresponds to a change of the Fuchsian group Γ. This provides a canonical identification between the Teichmüller space $T_{g,n}$ of Riemann surfaces of type (g,n) and the Teichmüller space of Fuchsian groups of type (g,n). Under this identification, the tangent space of $T_{g,n}$ at $[\Sigma]$ can be identified with the subspace $\Omega^{-1,1}(\Sigma) = \ker \overline{\partial}^*_{-1} \subset \mathcal{H}_{-1,1}$ of harmonic Beltrami differentials. Each element of $\mu \in \Omega^{-1,1}(\Sigma)$ has the form $\mu = y^2\overline{\varphi}$ for a unique $\varphi \in \ker \overline{\partial}_2$, so that $\dim_{\mathbb{C}} \Omega^{-1,1}(\Sigma) = 3g-3+n$. In particular, an element of $\Omega^{-1,1}(\Sigma)$ decays exponentially fast as one approaches a puncture (using the coordinates of (2.7)). The (holomorphic) cotangent space $T^*_{[\Sigma]}T_{g,n}$ can be identified with $\ker \overline{\partial}_2$ on Σ, this space being naturally dual to $\Omega^{-1,1}(\Sigma)$ via the pairing

$$(\mu,\varphi) := \int_\Sigma \mu\varphi, \quad \mu \in \Omega^{-1,1}(\Sigma),\ \varphi \in \ker \overline{\partial}_2. \tag{5.5}$$

To get complex coordinates on $T_{g,n}$ we can use the fact that to every $\mu \in \Omega^{-1,1}(\Sigma)$ satisfying

$$\|\mu\|_{L^\infty} = \sup_{z\in\Sigma} |\mu(z)| < 1, \tag{5.6}$$

one can associate a unique diffeomorphism $f^\mu : \mathbb{H} \to \mathbb{H}$ satisfying the Beltrami equation

$$\frac{\partial f^\mu}{\partial \overline{z}} = \mu \frac{\partial f^\mu}{\partial z} \tag{5.7}$$

and fixing the points $0, 1, \infty$, where μ in (5.7) is seen as an automorphic form of weight $(-2,2)$ on $\mathbb{H}$. From this solution, one gets a new Fuchsian group by considering $\Gamma^\mu := f^\mu \Gamma (f^\mu)^{-1}$, that is, a new complex structure by considering the Riemann surface $\Sigma^\mu := \Gamma^\mu \setminus \mathbb{H}$. The diffeomorphism f^μ also naturally descends to the quotient $\Gamma \setminus \mathbb{H}$ to give a diffeomorphism

$$f^\mu : \Gamma \setminus \mathbb{H} \to \Gamma^\mu \setminus \mathbb{H}. \tag{5.8}$$

Now, if one chooses a basis $\mu_1, \ldots, \mu_{3g-3+n}$ of $\Omega^{-1,1}(\Sigma)$ and sets $\mu = \varepsilon_1\mu_1 + \cdots + \varepsilon_{3g-3+n}\mu_{3g-3+n}$, then the correspondence $(\varepsilon_1, \ldots, \varepsilon_{3g-3+n}) \mapsto [\Sigma^\mu]$ defines complex coordinates in a neighborhood of $[\Sigma] \in T_{g,n}$ called **Bers coordinates**. In the overlapping of neighborhoods of two points $[\Sigma]$ and $[\Sigma^\mu]$, the Bers coordinates transform complex analytically (see for instance p.409 in [30]), defining on $T_{g,n}$ a complex structure. The Bers coordinates

provide a local trivialization of the fibration $p : \mathcal{T}_{g,n} \to T_{g,n}$ of the universal Teichmüller curve, in fact, of its universal cover, the Bers fibre space $\mathcal{BF}_{g,n}$ (see p.138 in [34]). If $\mathcal{U} \subset T_{g,n}$ is the open set where the Bers coordinates $(\varepsilon_1, \ldots, \varepsilon_{3g-g+n})$ associated to $[\Sigma]$ are defined, then this trivialization is given by the commutative diagram

$$\begin{tikzcd} \mathcal{U} \times \Sigma \arrow[r, "\nu"] \arrow[dr, "\mathrm{pr}_1"'] & p^{-1}(\mathcal{U}) \arrow[d, "p"] \\ & \mathcal{U} \end{tikzcd} \tag{5.9}$$

where pr_1 is the projection on the first factor and ν is given by $\nu(\mu, \sigma) = f^\mu(\sigma) \in p^{-1}([\Sigma^\mu])$ where f^μ denotes the map (5.8).

This local trivialization also induces a lift of $T_{[\Sigma]}T_{g,n}$ to $T\mathcal{T}_{g,n}|_{p^{-1}([\Sigma])}$, namely (see p.142 in [34]), a vector $\mu \in T_{[\Sigma]}T_{g,n}$ has a canonical lift $pr_1^*\mu \in T(\mathcal{U} \times \Sigma)|_{\{[\Sigma]\}\times\Sigma}$, and therefore a canonical lift $\nu_*(\mathrm{pr}_1^*\,\mu) \in T\mathcal{T}_{g,n}|_{p^{-1}([\Sigma])}$. More generally, introducing Bers coordinates at each $[\Sigma] \in T_{g,n}$, we can get in this way a canonical horizontal lift of $TT_{g,n}$ to $T\mathcal{T}_{g,n}$. In other words, associated to the fibration $p : \mathcal{T}_{g,n} \to T_{g,n}$, there is a **canonical connection** $\mathcal{P}$, that is, $\mathcal{P} \subset T\mathcal{T}_{g,n}$ is a distribution of hyperplanes such that

$$p_* : \mathcal{P}_z \to T_{p(z)}T_{g,n} \tag{5.10}$$

is an isomorphism for every $z \in \mathcal{T}_{g,n}$. It is also possible to define a covariant derivative

(5.11)

$$\nabla^{\mathcal{P}} : \mathcal{C}^\infty(\mathcal{T}_{g,n}; (\Lambda_v^{1,0})^\ell \otimes (\Lambda_v^{0,1})^m) \to \mathcal{C}^\infty(\mathcal{T}_{g,n}; p^*(T_{g,n}^*) \otimes (\Lambda_v^{1,0})^\ell \otimes (\Lambda_v^{0,1})^m).$$

This allows one to differentiate sections of $(\Lambda_v^{1,0})^\ell \otimes (\Lambda_v^{0,1})^m$ with respect to vectors on the base $T_{g,n}$. At $[\Sigma] \in T_{g,n}$, the differentiation can be described by using the Bers coordinates associated to $T_{[\Sigma]}T_{g,n} \cong \Omega^{-1,1}(\Sigma)$ with the local trivialization (5.9) of $p : \mathcal{T}_{g,n} \to T_{g,n}$ near $[\Sigma]$. In this trivialization, a section ω of $(\Lambda_v^{1,0})^\ell \otimes (\Lambda_v^{0,1})^m$ corresponds to a section $\widetilde{\omega}$ of $(\mathrm{pr}_2^*\,\Lambda_\Sigma^{1,0})^\ell \otimes (\mathrm{pr}_2^*\,\Lambda_\Sigma^{0,1})^m$ on $\mathcal{U} \times \Sigma$ where $\mathrm{pr}_2 : \mathcal{U} \times \Sigma \to \Sigma$ is the projection on the second factor. Precisely, in terms of automorphic forms of weight $(2\ell, 2m)$, we have that

$$\widetilde{\omega}(\varepsilon, \sigma) = \omega \circ f^\mu \left(\frac{\partial f^\mu}{\partial z}\right)^\ell \left(\overline{\frac{\partial f^\mu}{\partial z}}\right)^m \tag{5.12}$$

where $\mu = \varepsilon_1\mu_1 + \cdots + \varepsilon_{3g-3+n}\mu_{3g-3+n}$. On $\Sigma = p^{-1}([\Sigma]) \subset \mathcal{T}_{g,n}$, there is a canonical identification between $(\Lambda_v^{1,0})^\ell \otimes (\Lambda_v^{0,1})^m$ and $(\Lambda_\Sigma^{1,0})^\ell \otimes (\Lambda_\Sigma^{0,1})^m$. Under this identification, the covariant derivative of ω takes the form (cf. p.409 in [30]),

$$\nabla^{\mathcal{P}}_{\frac{\partial}{\partial \varepsilon_i}} \omega \Big|_{p^{-1}([\Sigma])} = \frac{\partial}{\partial \varepsilon_i} \widetilde{\omega}(\varepsilon, \sigma)\Big|_{\varepsilon=0}. \tag{5.13}$$

An important example is given by the family of fibrewise hyperbolic area forms dg_Σ, which as was shown in [1] gives a parallel section of $\Lambda_v^{1,1}$ with respect to the connection $\mathcal{P}$,

$$\nabla^{\mathcal{P}} dg_\Sigma = 0.$$

This corresponds to the fact that the automorphic form of weight $(2,2)$ $\frac{1}{y^2}$ is parallel with respect to the connection $\mathcal{P}$. However, notice that this does not imply the family of hyperbolic metrics $g_\Sigma, [\Sigma] \in T_{g,n}$ is parallel with respect to $\mathcal{P}$ as a section of $T_v^*\mathcal{T}_{g,n} \otimes T_v^*\mathcal{T}_{g,n}$. In fact, they cannot be parallel with respect to any connection, since otherwise this would mean that these metrics are all isometric, a contradiction since essentially by definition of the Teichmüller space, these metrics are not even conformal to one another.

It is also possible to define the covariant derivative of families of operators using the connection $\mathcal{P}$. If $A^\varepsilon : \mathcal{H}_{\ell,m}(\Sigma^\mu) \to \mathcal{H}_{\ell',m'}(\Sigma^\mu)$ is such family in the trivialization (5.9) given by the Bers coordinates, then the covariant derivative of A^ε at $[\Sigma]$ is given by

$$\begin{aligned} \nabla^{\mathcal{P}}_{\frac{\partial}{\partial \varepsilon_i}} A^\varepsilon \Big|_{[\Sigma]} &= \frac{\partial}{\partial \varepsilon_i} (f^\mu)^* A^\varepsilon (f^{\mu *})^{-1} \Big|_{\varepsilon=0}, \\ \nabla^{\mathcal{P}}_{\frac{\partial}{\partial \overline{\varepsilon}_i}} A^\varepsilon \Big|_{[\Sigma]} &= \frac{\partial}{\partial \overline{\varepsilon}_i} (f^\mu)^* A^\varepsilon (f^{\mu *})^{-1} \Big|_{\varepsilon=0}. \end{aligned} \tag{5.14}$$

For example, the covariant derivatives of $\overline{\partial}_\ell$ and $\overline{\partial}_\ell^*$ at $[\Sigma]$ are given by (see formula (2.6) in [30])

$$\begin{gathered} \nabla^{\mathcal{P}}_\mu \overline{\partial}_\ell = \mu \overline{\partial}^*_{\ell+1} u, \quad \nabla^{\mathcal{P}}_{\overline{\mu}} \overline{\partial}_\ell = 0, \\ \nabla^{\mathcal{P}}_\mu \overline{\partial}^*_\ell = 0, \quad \nabla^{\mathcal{P}}_{\overline{\mu}} \overline{\partial}^*_\ell = \overline{\mu} \overline{\partial}_{\ell-1} u^{-1} \end{gathered} \tag{5.15}$$

where $u := \frac{1}{y^2}$ is seen as a section of $\Lambda_\Sigma^{1,0} \otimes \Lambda_\Sigma^{0,1}$.

As we have seen, each Riemann surface Σ of type (g,n) has a boundary compactification Σ_{hc} constructed using the metric g_Σ. These compactifications fit together to give a fibrewise boundary compactification ${}^{\mathrm{hc}}\mathcal{T}_{g,n}$ of the universal Teichmüller curve. In terms of the local trivializations of (5.9), this is because the solution f^μ to the Beltrami equation (5.7) is real analytic (see for instance proposition 4.6.2 in [18]), it maps the fixed points of Γ to the fixed points of Γ^μ and, seen as a map $f^\mu : \Sigma \to \Sigma^\mu$, it is asymptotically holomorphic as one approaches any puncture of Σ. Since the canonical connection $\mathcal{P}$ is obtained by using Bers coordinates and infinitesimal deformations induced by the solutions of the Beltrami equation (5.7), we see that it also naturally lifts to provide a canonical connection ${}^{\mathrm{hc}}\mathcal{P}$ to the fibration

$${}^{\mathrm{hc}}p : {}^{\mathrm{hc}}\mathcal{T}_{g,n} \to T_{g,n}.$$

To get a natural boundary defining function for ${}^{\mathrm{hc}}\mathcal{T}_{g,n}$, we use the construction of (2.10) in each fibre. This definition depends on the choice of a number $\epsilon > 0$ which has to be chosen so that each cusp end N_i in a given surface has area strictly greater than ϵ. To get a global definition ${}^{\mathrm{hc}}\mathcal{T}_{g,n}$, we

should replace the number ϵ by a smooth function $a : T_{g,n} \to \mathbb{R}^+$ such that in a given fibre $\Sigma := p^{-1}([\Sigma])$, the area of each cusp end N_i is strictly greater than $a([\Sigma])$. We can then define our global defining function on ${}^{\mathrm{hc}}\mathcal{T}_{g,n}$ to be

$$\rho(\sigma) = \rho_{\Sigma,a([\Sigma])}(\sigma) \qquad \text{for} \quad \sigma \in \Sigma := p^{-1}([\Sigma]), \quad [\Sigma] \in T_{g,n} \tag{5.16}$$

where $\rho_{\Sigma,\epsilon} : \Sigma_{\mathrm{hc}} \to \mathbb{R}$ is defined in (2.10) for the Riemann surface Σ and a choice of small $\epsilon > 0$.

6. A local formula for the family index

The family of operators $\overline{\partial}_\ell \in \Psi^1(\mathcal{T}_{g,n}/T_{g,n}; {}^{\mathrm{hc}}K_v^\ell, {}^{\mathrm{hc}}\Lambda_v^{0,1} \otimes {}^{\mathrm{hc}}K_v^\ell)$ is a particular example of the families of ϕ-hc operators considered in [2]. When we apply this local index theorem to our family $\overline{\partial}_\ell$ with the canonical connection ${}^{\mathrm{hc}}\mathcal{P}$ for the fibration ${}^{\mathrm{hc}}p : {}^{\mathrm{hc}}\mathcal{T}_{g,n} \to T_{g,n}$, we get the family version of (3.9),

$$\begin{aligned} \mathrm{Ch}(\mathrm{Ind}(\overline{\partial}_\ell)) = \int_{\mathcal{T}_{g,n}/T_{g,n}} & \mathrm{Ch}(T_v^{-\ell}(\mathcal{T}_{g,n}))\,\mathrm{Td}(T_v\mathcal{T}_{g,n}) - \widehat{\eta}(D_\ell^V) \\ & - \widehat{\eta}(D_\ell^H) - \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{N}{2}} d\int_0^\infty \mathrm{Str}\left(\frac{\partial \mathbb{A}^t_{D_\ell}}{\partial t} e^{-(\mathbb{A}^t_{D_\ell})^2}\right) dt, \end{aligned} \tag{6.1}$$

where the eta invariants of the vertical and horizontal families are replaced by the the corresponding eta forms of Bismut and Cheeger [7] (with non-standard $\mathbb{Z}_2$ grading for D^H). This is an equality at the level of forms. Notice that in [2] the first term is expressed in terms of the $\widehat{A}$ form. However, thanks to Theorem 5.5 in [34] and its reformulation in equation 5.3 of [34], the fibration $p : \mathcal{T}_{g,n} \to T_{g,n}$ is Kähler fibration (see [10] for a defintion) so that it is possible to rewrite the first term using the Todd form instead. In the last term, $\mathbb{A}^t_{D_\ell}$ is the rescaled Bismut superconnection while N is the number operator in $\Lambda T_{g,n}$, that is, the action of N on forms of degree k on $T_{g,n}$ is multiplication by k.

Remark 6.1. *In this paper, our convention for the Chern character differs from that of* [5]. *This is why we need to include these extra factors of $2\pi i$ in the last term. In principle, the eta forms would also require such factors, so really, by an eta form, we mean $(2\pi i)^{-\frac{N}{2}}$ times the eta form of Bismut and Cheeger (cf. equation 4.101 in* [7]*).*

When we take the degree zero part of (6.1), we get back the numerical index (3.9) by evaluating it at a given point $[\Sigma] \in T_{g,n}$. In fact, as we have seen, the degree zero part of $\widehat{\eta}(D_\ell^V)$ is identically zero, while the degree zero part of $\widehat{\eta}(D_\ell^H)$ is $\frac{n}{2}\,\mathrm{sign}(\ell - \frac{1}{2})$. However, the higher degree components of $\widehat{\eta}(D_\ell^H)$ vanish identically at the level of forms as we will see in a moment.

Let $\partial\mathcal{T}_{g,n} := \rho^{-1}(0)$ be the union of the boundaries of the fibres of ${}^{\mathrm{hc}}p : {}^{\mathrm{hc}}\mathcal{T}_{g,n} \to T_{g,n}$. The map ${}^{\mathrm{hc}}p$ induces a fibration structure

$$\partial p : \partial\mathcal{T}_{g,n} \to T_{g,n} \tag{6.2}$$

with typical fibre the disjoint union of n circles. In fact, the manifold $\partial\mathcal{T}_{g,n}$ has precisely n components,

$$\partial\mathcal{T}_{g,n} = \bigcup_{i=1}^{n} \partial_i \mathcal{T}_{g,n} \tag{6.3}$$

with $\partial_i\mathcal{T}_{g,n}$ the component associated to the ith cusp. There is a corresponding fibration structure

$$\partial p_i : \partial_i\mathcal{T}_{g,n} \to T_{g,n}. \tag{6.4}$$

Recall that the vertical family D^V_ℓ decomposes as

$$D^V_\ell = \begin{pmatrix} 0 & D^{V,-}_\ell \\ D^{V,+}_\ell & 0 \end{pmatrix} \tag{6.5}$$

with respect to the $\mathbb{Z}_2$ grading of the Clifford bundle.

Lemma 6.2. *The Chern form of* $\ker D^{V,+}_\ell \to T_{g,n}$ *vanishes in positive degrees,*

$$\mathrm{Ch}(\ker D^{V,+}_\ell)_{[2k]} = 0, \quad k \in \mathbb{N}.$$

Proof. Let $D^V_{\ell,i}$ be the vertical family of the ith component $\partial_i\mathcal{T}_{g,n}$ of $\partial\mathcal{T}_{g,n}$. Via the identification

$$-c(\frac{d\rho}{\rho}) : {}^{\mathrm{hc}}\Lambda^{0,1}_v \otimes {}^{\mathrm{hc}}K_v \longrightarrow {}^{\mathrm{hc}}K_v$$

given by Clifford multiplication, the operator $D^{V,+}_\ell$ can be identified with

$$\frac{1}{i}\nabla_{\frac{\partial}{\partial x}} = i\nabla_{\frac{\partial}{\partial u}} : \mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}; {}^{\mathrm{hc}}K^\ell_v) \to \mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}; {}^{\mathrm{hc}}K^\ell_v) \tag{6.6}$$

where $u = -x$ is such that $\frac{\partial}{\partial u}$ is an oriented orthonormal basis of $T_\sigma(\partial_i\mathcal{T}_{g,n}/T_{g,n})$ for each $\sigma \in \partial_i\mathcal{T}_{g,n}$. Thus, $\ker D^{V,+}_{\ell,i} \to T_{g,n}$ defines a complex line bundle over the Teichmüller space $T_{g,n}$ and

$$\ker D^{V,+}_\ell = \bigoplus_{i=1}^{n} \ker D^{V,+}_{\ell,i}. \tag{6.7}$$

For the corresponding Chern characters, this gives

$$\mathrm{Ch}(\ker D^{V,+}_\ell) = \sum_{i=1}^{n} \mathrm{Ch}(\ker D^{V,+}_{\ell,i}). \tag{6.8}$$

To prove the lemma, it therefore suffices to show that $\mathrm{Ch}(\ker D^{V,+}_{\ell,i})_{[2]} = 0$ for $i \in \{1, \ldots, n\}$. This will be true provided we can trivialize $\ker D^{V,+}_{\ell,i}$ by

a parallel section. From the identification of $D^{V,+}_{\ell,i}$ with (6.6), a choice of trivializing section is given by taking

$$s_{\ell,i} : [\Sigma] \mapsto \left(\rho dx - i\frac{d\rho}{\rho}\right)^{\ell}\Bigg|_{(\partial\Sigma_{\mathrm{hc}})_i} \in {}^{\mathrm{hc}}K_v^{\ell}\Big|_{(\partial\Sigma_{\mathrm{hc}})_i} \tag{6.9}$$

where $\Sigma = p^{-1}([\Sigma])$ and ρ is the boundary defining function of (5.16). Notice that the section (6.9) is completely determined by the canonical family of hyperbolic metrics $g_{\mathcal{T}_{g,n}/T_{g,n}}$. Conversely, for $\ell = 1$, the section $s_{1,i}$ completely determines the asymptotic behavior of $g_{\mathcal{T}_{g,n}/T_{g,n}}$ as one approaches the i^{th} puncture. If the family of metrics $g_{\mathcal{T}_{g,n}/T_{g,n}}$ were parallel with respect to the canonical connection $\mathcal{P}$, we could conclude immediately that the section $s_{\ell,i}$ is parallel. This is not the case, but at least the family of metrics $g_{\mathcal{T}_{g,n}/T_{g,n}}$ is asymptotically parallel as one approaches a puncture. Indeed, from (5.15), we see that the parallel transport (along a path on $T_{g,n}$) defined by the canonical connection $\mathcal{P}$ is asymptotically holomorphic as one approaches a puncture. This is because the Beltrami differential μ in (5.7) vanishes exponentially fast as one approaches a puncture (using the coordinates of (2.7)). Thus, parallel transport is asymptotically a conformal transformation for the family of metrics $g_{\mathcal{T}_{g,n}/T_{g,n}}$. Since

$$\nabla^{\mathcal{P}} dg_{\mathcal{T}_{g,n}/T_{g,n}} = 0, \tag{6.10}$$

this means that the parallel transport defined by the connection $\mathcal{P}$ is asymptotically an isometry as one approaches a puncture. That is, $\nabla^{\mathcal{P}} g_{\mathcal{T}_{g,n}/T_{g,n}}$ is asymptotically zero as one approaches a puncture. In particular, this implies that for each $i \in \{1, \ldots, n\}$, the section $s_{\ell,i}$ of (6.9) is parallel with respect to the connection $\mathcal{P}$.

$\square$

Together with the boundary defining function ρ, the family of metric $g_{\mathcal{T}_{g,n}/T_{g,n}}$ induces a natural family of metrics g_i for each fibre of the fibration (6.4) in such a way that each fibre becomes isometric to the circle $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ of length 1 (cf. [35]). With these identifications, we get a natural action of $\mathbb{S}^1$ on each fibre, giving (6.4) the structure of a principal $\mathbb{S}^1$-bundle. By construction, the family of metrics g_i is $\mathbb{S}^1$-equivariant with respect to the $\mathbb{S}^1$ action. The canonical connection ${}^{\mathrm{hc}}\mathcal{P}$ naturally induces a connection $\mathcal{P}_i$ on (6.4).

Lemma 6.3. *The family of metrics g_i is parallel with respect to the connection $\mathcal{P}_i$, that is, the connection $\mathcal{P}_i$ is unitary with respect to the metric g_i. In particular, on the ith circular boundary component, the vector field $\frac{\partial}{\partial u}$ is parallel with respect to the connection $\mathcal{P}_i$.*

Proof. By the proof of lemma 6.2, the family of metrics $g_{\mathcal{T}_{g,n}/T_{g,n}}$ is asymptotically parallel as one approaches a cusp, from which the result follows. $\square$

We can now show that the eta form of D^H_{ℓ} vanishes in positive degrees.

Lemma 6.4. *For each $k \in \mathbb{N}$, the degree $2k$ part of the form $\widehat{\eta}(D_\ell^H)$ vanishes identically,*

$$\widehat{\eta}(D_\ell^H)_{[2k]} = 0, \quad k > 0.$$

Proof. Since D_ℓ^H is just an endomorphism of $\ker D_\ell^V$, we see from proposition 3.1, lemma 6.3 and the definition of the eta form that (see [2, (4.12)])

$$\widehat{\eta}(D_\ell^H) = \frac{1}{2}\operatorname{sign}\left(\ell - \frac{1}{2}\right)\operatorname{Ch}(\ker D_\ell^{V,+}). \tag{6.11}$$

The result then follows from lemma 6.2. □

On the other hand, the eta form of the vertical family gives a contribution in higher degrees. In fact, since the geometry of the boundary fibration is very special, it is possible to compute the eta form explicitly. With respect to the decomposition (6.3), the vertical family D_ℓ^V admits a corresponding decomposition

$$D_\ell^V = \bigoplus_{i=1}^n D_{\ell,i}^V \tag{6.12}$$

where $D_{\ell,i}^V$ is a family of self-adjoint Dirac operators on the fibration (6.4). In terms of this decomposition, the eta form of D_ℓ^V can be expressed as

$$\widehat{\eta}(D_\ell^V) = \sum_{i=1}^n \widehat{\eta}(D_{\ell,i}^V). \tag{6.13}$$

By (5.15) (see also the proof of lemma 8.1), the family of Dirac operators D_ℓ is asymptotically parallel with respect to the canonical connection $\mathcal{P}$ as one approaches a cusp. This means that each of the vertical families $D_{\ell,i}^V$ is parallel with respect to the connection $\mathcal{P}_i$ on (6.4). This fact, together with the fact the family of metric g_i is parallel with respect to the connection $\mathcal{P}_i$ and is equivariant with respect to the circle action, means that we can apply the result of Zhang (Theorem 1.7 in [36]) to get an explicit formula for the eta form $\eta(D_{\ell,i}^V)$.

Proposition 6.5 (Zhang, [36], Theorem 1.7)**.** *The eta form of $D_{\ell,i}^V$ is given by*

$$\widehat{\eta}(D_{\ell,i}^V) = \frac{1}{2\tanh\left(\frac{e_i}{2}\right)} - \frac{1}{e_i}$$

where $e_i := \frac{\sqrt{-1}}{2\pi}\Theta_i$ is the curvature form of the circle bundle $\partial p_i : \partial_i \mathcal{T}_{g,n} \to T_{g,n}$ with connection $\mathcal{P}_i$ and curvature Θ_i, the Lie algebra of $\mathbb{S}^1$ being identified with $i\mathbb{R}$.

Remark 6.6. *Notice in particular that this implies that the eta form is zero in degree $2k$ for $k = 0$ modulo 2. Moreover, it is a closed form, an unusual feature for a eta form.*

Before stating our main theorem, let us give an alternate description of the Chern form e_i. Namely, to the circle bundle (6.4) with connection $\mathcal{P}_i$ and family of metrics $(2\pi)g_i$, we can associate in a canonical way a complex line bundle $\mathcal{L}_i \to T_{g,n}$ equipped with a Hermitian metric h_i and a unitary connection $\nabla^{\mathcal{L}_i}$ in such a way that the curvature form of $\mathcal{L}_i$ is precisely $(-2\pi\sqrt{-1})e_i$. The line bundle $\mathcal{L}_i$ is such that its unit circle bundle with induced metric and connection is precisely the circle bundle (6.4) with family of metrics $2\pi g_i$ and connection $\mathcal{P}_i$.

Thinking of a fibre $\Sigma := p^{-1}([\Sigma])$ as a punctured Riemann surface

$$\Sigma = \overline{\Sigma} - \{x_1, \ldots, x_n\}, \tag{6.14}$$

one can also define the line bundle $\mathcal{L}_i$ by

$$\mathcal{L}_{i,[\Sigma]} := (T^{1,0}_{x_i}\overline{\Sigma})^* = K_{\overline{\Sigma}}\big|_{x_i}, \quad [\Sigma] \in T_{g,n}. \tag{6.15}$$

Moreover, from this perspective, the Hermitian metric h_i and the unitary connection $\nabla^{\mathcal{L}_i}$ are easily seen to be the same as the one introduced by Wolpert [35]. Thus, the form e_i corresponds to the Chern form $c_1(\| \,\|_{can,i})$ of Corollary 7 in [35].

Now, combining (6.1) with Lemma 6.4 and Proposition 6.5, we obtain the following formula.

Theorem 1. *The local family index of the family of operators*

$$D^+_\ell := \sqrt{2}\,\overline{\partial}_\ell \in \rho^{-1}\Psi^1_{\mathrm{cu}}(\mathcal{T}_{g,n}/T_{g,n}; {}^{\mathrm{hc}}K^\ell_v, {}^{\mathrm{hc}}\Lambda^{0,1}_v \otimes {}^{\mathrm{hc}}K^\ell_v)$$

associated to the Teichmüller universal curve $p : \mathcal{T}_{g,n} \to T_{g,n}$ and its canonical connection $\mathcal{P}$ is given by

(6.16)

$$\begin{aligned}\mathrm{Ch}(\operatorname{ind}\ker D^+_\ell) &= \int_{\mathcal{T}_{g,n}/T_{g,n}} \mathrm{Ch}(T_v^{-\ell}(\mathcal{T}_{g,n}))\,\mathrm{Td}(T_v\mathcal{T}_{g,n}) + \frac{n}{2}\operatorname{sign}\left(\frac{1}{2} - \ell\right) \\ &- \sum_{i=1}^n \left(\frac{1}{2\tanh\left(\frac{e_i}{2}\right)} - \frac{1}{e_i}\right) - \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{N}{2}} d\int_0^\infty \mathrm{Str}\left(\frac{\partial \mathbb{A}^t_{D_\ell}}{\partial t} e^{-(\mathbb{A}^t_{D_\ell})^2}\right) dt,\end{aligned}$$

where $\mathbb{A}^t_{D_\ell}$ is the rescaled Bismut superconnection associated to the family D_ℓ, e_i is the canonical Chern form of the (holomorphic) cotangent bundle along the i^{th} cusp $\mathcal{L}_i \to T_{g,n}$ and N is the number operator on $\Lambda T_{g,n}$.

As in [30], each of the terms in our formula is invariant under the action of the Teichmüller modular group $\mathrm{Mod}_{g,n}$. Thus, formula (6.16) also holds on the moduli space $\mathcal{M}_{g,n} := T_{g,n}/\mathrm{Mod}_{g,n}$ in the sense of orbifolds with the fibration $p : \mathcal{T}_{g,n} \to T_{g,n}$ replaced by the forgetful map $\pi_{n+1} : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$. In fact, on the moduli space $\mathcal{M}_{g,n}$, the formula acquires a topological meaning in higher degrees.

To see this, define $\overline{\mathcal{T}}_{g,n}$ to be the space obtained from $\mathcal{T}_{g,n}$ by filling each puncture of each fibre by a marked point. There is still a fibration $\overline{p} : \overline{\mathcal{T}}_{g,n} \to T_{g,n}$, but now with fibres being compact Riemann surfaces of

genus g with n marked points. Let $\overline{K}_v \to \mathcal{T}_{g,n}$ denote the corresponding vertical canonical line bundle (the dual of the vertical $(1,0)$ tangent bundle). Let $D_i \subset \overline{\mathcal{T}}_{g,n}$ be the divisor associated to the ith marked points and let L_D be the line bundle associated to the divisor $D := \sum_{i=1}^n D_i$. Then, by analogy with the discussion in § 3, we see that the family index of $\overline{\partial}_\ell$ is the same as the family index of the family of $\overline{\partial}$-operators

$$\widehat{\overline{\partial}}_\ell : \mathcal{C}^\infty(\mathcal{T}_{g,n}; \overline{K}_v^\ell \otimes L_D^{\ell-1}) \to \mathcal{C}^\infty(\mathcal{T}_{g,n}; \Lambda_v^{0,1} \otimes \overline{K}_v^\ell \otimes L_D^{\ell-1}) \tag{6.17}$$

for $\ell > 0$ and

$$\widehat{\overline{\partial}}_\ell : \mathcal{C}^\infty(\mathcal{T}_{g,n}; \overline{K}_v^\ell \otimes L_D^{\ell}) \to \mathcal{C}^\infty(\mathcal{T}_{g,n}; \Lambda_v^{0,1} \otimes \overline{K}_v^\ell \otimes L_D^{\ell}) \tag{6.18}$$

for $\ell \leq 0$. On the fibration $\pi_{n+1} : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$, this corresponds to the following situation. Let $\omega_{\pi_{n+1}}$ be the relative dualizing sheaf of this fibration, that is, the sheaf of sections of $\overline{K}_v$. Let $\omega_{\pi_{n+1}}(D)$ be the logarithmic variant of $\omega_{\pi_{n+1}}$, which means that the local sections of $\omega_{\pi_{n+1}}(D)$ are sections of $\omega_{\pi_{n+1}}$ with possibly simple poles at the first n marked points. Then the line bundle $\overline{K}_v \otimes L_D$ on $\mathcal{T}_{g,n}$ corresponds to the sheaf $\omega_{\pi_{n+1}}(D)$ on $\mathcal{M}_{g,n+1}$.

Going back to the formula of theorem 1, we see that the form e_i then represents the Miller class $\psi_i = c_1(\mathcal{L}_i)$. On the other hand, since the Miller class ψ_{n+1} on $\mathcal{M}_{g,n+1}$ is given by $\psi_{n+1} = c_1(\omega_{\pi_{n+1}}(D))$ (see for instance p.254 in [33]), the first term in the right-hand side of (6.16) can be seen to represent a linear combination of the Mumford-Morita classes

$$\kappa_j := (\pi_{n+1})_*(\psi_{n+1}^{j+1}) = \left[(\pi_{n+1})_*(e_{n+1}^{j+1})\right], \quad j \in \mathbb{N}_0, \tag{6.19}$$

where e_{n+1} is the Chern form of the vertical canonical line bundle $K_v \cong \overline{K}_v \otimes L_D$. The precise formula involves the Bernouilli numbers B_m and the Bernouilli polynomials $B_m(\ell)$, which are defined by the following identities,

$$\frac{x}{e^x - 1} = \sum_{m \geq 0} B_m \frac{x^m}{m!}, \quad \frac{e^{\ell x} x}{e^x - 1} = \sum_{m \geq 0} B_m(\ell) \frac{x^m}{m!}. \tag{6.20}$$

Thus, the first term in (6.16) is seen to represent the cohomology class

$$(\pi_{n+1})_* \left(\frac{e^{\ell \psi_{n+1}} \psi_{n+1}}{e^{\psi_{n+1}} - 1} \right) = \sum_{m \geq 1} B_m(\ell) \frac{\kappa_{m-1}}{m!}. \tag{6.21}$$

On the moduli space, theorem 1 therefore gives the following local formula (in the sense of orbifolds).

Corollary 6.7. *In the sense of orbifolds, the Chern character of the index of the family $\overline{\partial}_\ell$ associated to the forgetful map $\pi_{n+1}\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ is given*

at the form level by

$$\operatorname{Ch}(\ker D_\ell^+) - \operatorname{Ch}(\ker D_\ell^-) = \sum_{m\geq 1} \frac{B_m(\ell)}{m!} k_{m-1} + \frac{n}{2}\operatorname{sign}\left(\frac{1}{2}-\ell\right) \tag{6.22}$$

$$-\sum_{i=1}^n \left(\frac{1}{2\tanh\left(\frac{e_i}{2}\right)} - \frac{1}{e_i}\right) - \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{N}{2}} d\int_0^\infty \operatorname{Str}\left(\frac{\partial \mathbb{A}^t_{D_\ell}}{\partial t} e^{-(\mathbb{A}^t_{D_\ell})^2}\right) dt,$$

where $k_m = (\pi_{n+1})_*(e_{n+1}^{m+1})$ *and* e_i *are canonical form representatives of the Morita-Mumford-Miller classes* κ_m *and* ψ_i*.*

If $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford compactification of the moduli space $\mathcal{M}_{g,n}$, then theorem 1 can be intuitively interpreted as a local version of the Grothendieck-Riemann-Roch theorem applied to the morphism $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ and the sheaf

$$\widetilde{\omega}_\ell := \begin{cases} \omega_{\pi_{n+1}}(D)^{\ell-1} \otimes \omega_{\pi_{n+1}}, & \ell > 0, \\ \omega_{\pi_{n+1}}(D)^\ell, & \ell \leq 0. \end{cases} \tag{6.23}$$

In this context, the Grothendieck-Riemann-Roch theorem was first studied and used by Mumford [27] in the case $n = 0$ with formula given by

$$\operatorname{Ch}((\pi_1)_*\omega_{\pi_1}^\ell) = \sum_{m\geq 1} \frac{B_m(\ell)}{m!}\kappa_{m-1} + (\text{terms coming from } \partial\overline{\mathcal{M}}_g).$$

When $n > 0$, a Grothendieck-Riemann-Roch formula was obtained for the sheaf $\omega_{\pi_{n+1}}^\ell$ by Bini [6],

$$\operatorname{Ch}((\pi_{n+1})_*\omega_{\pi_{n+1}}^\ell) = \sum_{m\geq 1} \frac{B_m(\ell)}{m!}\widetilde{\kappa}_{m-1} + (\text{terms coming from } \partial\overline{\mathcal{M}}_{g,n}). \tag{6.24}$$

where $\widetilde{\kappa}_m := (\pi_{n+1})_*\left(c_1(\omega_{\pi_{n+1}})^{m+1}\right)$. When $\ell = 0$ and $\ell = 1$, it makes sense to compare our formula with the one of Bini. In that case, using the relation

$$\kappa_m = \widetilde{\kappa}_m + \sum_{i=1}^n \psi_i^m$$

proved by Arbarello and Cornalba [4] together with the identity

$$\frac{x}{2\tanh\frac{x}{2}} = \frac{x}{e^x - 1} + \frac{x}{2},$$

we can easily check that, as expected, our formula agrees with the interior contribution of (6.24).

7. The spectral hc-zeta determinant

On any geometrically finite hyperbolic surface $\Sigma = \Gamma \setminus \mathbb{H}$, the Selberg's zeta

function is defined for $\mathrm{Re}\,(s) > 1$ to be

$$Z_\Sigma\,(s) = \prod_{\{\gamma\}} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell(\gamma)}\right) \tag{7.1}$$

where the outer product goes over conjugacy classes of primitive *hyperbolic* elements of Γ and $\ell\,(\gamma)$ is the length of the corresponding closed geodesic.

On closed hyperbolic surfaces, a well-known result of D'Hoker and Phong [13] says that the determinant of the Laplacian $\Delta_{\Sigma,\ell}$ acting on sections of K^ℓ can be expressed in terms of special values of the Selberg's Zeta function,

$$\begin{aligned} \det(\Delta_{\Sigma,\ell}) &= Z_\Sigma(\ell) e^{-c_{\ell-1}\chi(\Sigma)}, \quad \ell \geq 2, \\ \det{}'(\Delta_{\Sigma,\ell}) &= Z'_\Sigma(1) e^{-c_0\chi(\Sigma)}, \quad\ \ \ell = 0, 1. \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} c_\ell := \sum_{0\leq m<\ell-\frac{1}{2}} (2\ell - 2m - 1)\log(2\ell - m) - \left(\ell + \frac{1}{2}\right)^2 \\ + \left(\ell + \frac{1}{2}\right)\log 2\pi + 2\zeta_{Riem}(-1). \end{aligned} \tag{7.3}$$

Shortly after, it was shown by Sarnak [29] that for the geometric Laplacian with non-negative spectrum Δ_Σ,

$$\frac{\det\left(\Delta_\Sigma + s\,(s-1)\right)}{Z_\Sigma\,(s)} = \left(e^{E-s(s-1)}\frac{\Gamma_2\,(s)^2}{\Gamma\,(s)}\,(2\pi)^s\right)^{-\chi(\Sigma)} \tag{7.4}$$

where $E = -\frac{1}{4} - \frac{1}{2}\log 2\pi + 2\zeta'_{Riem}\,(-1)$, Γ_2 is the Barnes double Gamma function. As indicated in [29], the formula of D'Hoker and Phong can be recovered relatively easily from (7.4).

On a Riemann surface with cusps, the Selberg Zeta function as defined above still makes sense. However, since the Laplacian has a continuous spectrum, the definition of its determinant is more subtle. It was studied by Efrat [14], [15] and by Müller [26] using scattering theory to understand the contribution from the continuous spectrum. In this paper, we use renormalized integrals to extend the usual definition of the determinant via zeta-regularization to these manifolds, with the advantage that this does not require the metric to have constant curvature. We then use the analysis of [12] to show that, on hyperbolic surfaces, our definition satisfies (7.4) with the right-hand-side replaced by a meromorphic function depending only on the genus and the number of punctures, an important feature for our purposes.

7.1. **The determinant of the Laplacian.**

To relate the determinant with the Selberg Zeta function and get an analog of formula (7.4), it is convenient to work first with the (positive)

geometric Laplacian Δ_Σ instead of the $\overline{\partial}$-Laplacian. Recall that the two are the same modulo a multiplicative constant,

$$\Delta_{\overline{\partial}} = \frac{1}{2}\Delta_\Sigma.$$

Following [22, §9.5] and [17, §3], we define the zeta function of Δ_Σ using the renormalized trace (see e.g., [2])

$$\zeta_{\Delta_\Sigma}(z) := \frac{1}{\Gamma(z)}\int_0^\infty t^z \, {}^R\operatorname{Tr}\left(e^{-t\Delta_\Sigma} - \mathcal{P}_{\ker\Delta_\Sigma}\right)\, \frac{dt}{t}.$$

Since Δ_Σ is Fredholm, zero is spectrally isolated and the integrand decays exponentially for large times. Thus the integral defines a holomorphic function for $\operatorname{Re}(z)$ large enough. The small-times asymptotics of the integrand (whose existence follows from the construction of the heat kernel in [31]) allow us to extend the function meromorphically to the whole complex plane. We denote the meromorphic extension by the same symbol and define

$$\log\det\Delta_\Sigma := -\zeta'_{\Delta_\Sigma}(0)\,.$$

We can find a more explicit expression for the zeta function by subtracting the first few terms in the expansion of the heat kernel at $t=0$. The form of this expansion can be deduced for arbitrary ϕ-hc operators of Laplace-type from Vaillant's construction (see the appendix of [3] for such an approach), but for the case at hand the expansion is well-known (see, e.g., (2.3) in [26])

$$ {}^R\operatorname{Tr}(e^{-t\Delta_\Sigma}) = \frac{a_{-1}}{t} + \widetilde{a}_{-\frac{1}{2}}\frac{\log t}{\sqrt{t}} + \frac{a_{-\frac{1}{2}}}{\sqrt{t}} + a_0 + \mathcal{O}(\sqrt{t}) \quad \text{as } t\to 0^+. \tag{7.5}$$

Thus, writing $f_0(t) = a_{-1}t^{-1} + \widetilde{a}_{-\frac{1}{2}}\frac{\log t}{\sqrt{t}} + a_{-\frac{1}{2}}t^{-\frac{1}{2}} + a_0$ and choosing any $C>0$, we have the expression

$$\begin{aligned}\zeta_{\Delta_\Sigma}(z) &= \tfrac{1}{\Gamma(z)}\textstyle\int_0^C t^z\left({}^R\operatorname{Tr}\left(e^{-t\Delta_\Sigma}\right) - f_0(t)\right)\,\frac{dt}{t}\\ &+\tfrac{1}{\Gamma(z)}\textstyle\int_C^\infty t^z\left({}^R\operatorname{Tr}\left(e^{-t\Delta_\Sigma}\right) - \dim\ker_-(\Delta_\Sigma)\right)\,\frac{dt}{t}\\ &+\tfrac{C^z}{\Gamma(z+1)}\left(a_0 - \dim\ker_-(\Delta_\Sigma)\right) + \tfrac{C^{z-\frac{1}{2}}a_{-\frac{1}{2}}}{(z-\frac{1}{2})\Gamma(z)}\\ &+\tfrac{\widetilde{a}_{-\frac{1}{2}}C^{z-\frac{1}{2}}}{\Gamma(z)}\left(\tfrac{\log C}{z-\frac{1}{2}} - \tfrac{1}{(z-\frac{1}{2})^2}\right) + \tfrac{C^{z-1}a_{-1}}{(z-1)\Gamma(z)}.\end{aligned}$$

Differentiating and setting $z=0$, we get

$$\begin{aligned}\zeta'_{\Delta_\Sigma}(0) &= \int_0^C \left({}^R\operatorname{Tr}\left(e^{-t\Delta_\Sigma}\right) - f_0(t)\right)\,\frac{dt}{t}\\ &+\int_C^\infty \left({}^R\operatorname{Tr}\left(e^{-t\Delta_\Sigma}\right) - \dim\ker_-(\Delta_\Sigma)\right)\,\frac{dt}{t}\\ &+(\log C + \gamma_e)\left(a_0 - \dim\ker_-(\Delta_\Sigma)\right) - 2C^{-\frac{1}{2}}a_{-\frac{1}{2}}\\ &- C^{-1}a_{-1} + \widetilde{a}_{-\frac{1}{2}}C^{-\frac{1}{2}}(-4 - 2\log C)\end{aligned} \tag{7.6}$$

by using the fact that $\frac{1}{\Gamma(z)}\rfloor_{z=0}=0$, $\partial_z \rfloor_{z=0} \frac{1}{\Gamma(z)}=1$ and $\partial_z \rfloor_{z=0} \frac{1}{\Gamma(z+1)}=\gamma_e$ is Euler's gamma constant.

More generally, and to connect with (7.4), we can consider the determinant of Δ_Σ with its spectrum shifted by a complex number w, that is, the determinant of $\Delta_\Sigma + w$. Just as before we have

$$
\begin{aligned}
\zeta_{\Delta_\Sigma}(z;w) &= \frac{1}{\Gamma(z)}\int_0^\infty t^z \left({}^R\operatorname{Tr}\left(e^{-t\Delta_\Sigma}\right) - f_0(t)\right) e^{-tw}\,\frac{dt}{t} \\
&\quad + \frac{a_0}{w^z} + \frac{\widetilde{a}_{-\frac12}}{w^{z-\frac12}} \frac{\left(\Gamma_{\log}\left(z-\frac12\right) - \log w\,\Gamma\left(z-\frac12\right)\right)}{\Gamma(z)} \\
&\quad + \frac{a_{-\frac12}}{w^{z-\frac12}}\frac{\Gamma\left(z-\frac12\right)}{\Gamma(z)} + \frac{a_{-1}}{w^{z-1}}(z-1)^{-1}
\end{aligned}
\tag{7.7}
$$

where the function $\Gamma_{\log}(z)$ is defined to be

$$
\Gamma_{\log}(z) := \int_0^\infty t^z e^{-t}\log t\,\frac{dt}{t}
\tag{7.8}
$$

for $\operatorname{Re} z > 0$. Since it satisfies the recurrence relation

$$
\Gamma_{\log}(z+1) = z\Gamma_{\log}(z) + \Gamma(z),
$$

it has a meromorphic continuation to the whole complex plane with poles at $-\mathbb{N}_0 = 0,-1,-2\ldots$. In particular, it has no pole at $z=-\frac12$. Taking the derivative of $\zeta_{\Delta_\Sigma}(z;w)$ with respect to z and setting $z=0$, we get

$$
\begin{aligned}
\zeta'_{\Delta_\Sigma}(0;w) &= -\log\det\left(\Delta_\Sigma + w\right) \\
&= \int_0^\infty \left({}^R\operatorname{Tr}\left(e^{-t\Delta_\Sigma}\right) - f_0(t)\right) e^{-tw}\,\frac{dt}{t} \\
&\quad - a_0\log w - 2\sqrt{\pi}a_{-\frac12}\sqrt{w} + a_{-1}w\left(-1+\log w\right) \\
&\quad + \widetilde{a}_{-\frac12}\sqrt{w}\left(\Gamma_{\log}(-\frac12) - \log w\,\Gamma(-\frac12)\right).
\end{aligned}
\tag{7.9}
$$

7.2. **Relation with the Selberg Zeta function.**

To relate the determinant with the Selberg Zeta function, we will follow [12] and use a description of the Selberg Zeta function in terms of the resolvent of the Laplacian. Given a hyperbolic surface Σ of genus g with n cusps, we denote by

$$
R_\Sigma(s) := \left(\Delta_\Sigma + s(s-1)\right)^{-1}
\tag{7.10}
$$

the resolvent of the geometric Laplacian Δ_Σ with respect to the hyperbolic metric. The Schwartz kernel of $R_\Sigma(s)$ is singular along the diagonal. A natural way to remove this singular part is to subtract the resolvent of the model hyperbolic space

$$
R_{\mathbb{H}}(s) = \left(\Delta_{\mathbb{H}} + s(s-1)\right)^{-1}.
\tag{7.11}
$$

This resolvent has Schwartz kernel defined on $\mathbb{H}\times\mathbb{H}$. Hence, thinking of Σ as the quotient $\Gamma\setminus\mathbb{H}$ of the hyperbolic half-plane by some appropriate

discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$, there is a natural lift of the Schwartz kernel $G_\Sigma(s;z,w)$ of $R_\Sigma(s)$ to $\mathbb{H}\times\mathbb{H}$. Since locally $R_\Sigma(s)$ and $R_{\mathbb{H}}(s)$ have the same full symbol (being a parametrix for $\Delta_{\mathbb{H}} + s(s-1)$), they will have the same singularities along the diagonal. This means the function

$$\varphi_\Sigma(s;z) := (2s-1)\left[G_\Sigma(s;z,w) - G_{\mathbb{H}}(s;z,w)\right]_{w=z} \tag{7.12}$$

will be smooth in $z \in \mathbb{H}$ and meromorphic in s. Because of the $\mathrm{SL}(2,\mathbb{R})$ invariance of $\Delta_{\mathbb{H}}$, the Schwartz kernel $G_{\mathbb{H}}$ will also be $\mathrm{SL}(2,\mathbb{R})$ invariant when restricted to the diagonal in $\mathbb{H}\times\mathbb{H}$. This means in particular that the function $\varphi_\Sigma(s;z)$ will be Γ-invariant and so will descend to give a function on $\Sigma = \Gamma\setminus\mathbb{H}$. Thus, we can define the function

$$\phi_\Sigma(s) := {}^R\!\!\int_\Sigma \varphi_\Sigma(s;z)\,dg_\Sigma(z) \tag{7.13}$$

where dg_Σ is the volume form associated to the hyperbolic metric, and the integral is renormalized using ρ_Σ, the boundary defining function of (2.10). This function has a meromorphic continuation to the complex plane with possible poles at $\frac{1}{2} - \mathbb{N}_0/2$. As a particular example, we can consider the Horn $H := \Gamma_\infty \setminus \mathbb{H}$ of (2.4). The end obtained as $y \to +\infty$ is a cusp end, so we can pick a boundary defining function as usual there, but the other end when $y \to 0^+$ is not a cusp end, but a funnel, and for that end, one should take a boundary defining function which is given by y near $y=0$. With this choice, we can make sense of the function $\phi_H(s)$. The following proposition is due to Borthwick, Judge, and Perry [12].

Proposition 7.1 ([12], Proposition 4.3)**.** *Let Σ be a Riemann surface of genus g and with n cusps. Then*

$$\frac{Z'_\Sigma(s)}{Z_\Sigma(s)} = \phi_\Sigma(s) - n\phi_H(s).$$

The function $\phi_H(s)$ can be computed explicitly [12, Proposition 2.4],

$$\phi_H(s) = -\log 2 - \Psi\left(s + \frac{1}{2}\right) + \frac{1}{2s-1} \tag{7.14}$$

where $\Psi(z)$ is the digamma function $\frac{\Gamma'(z)}{\Gamma(z)}$. Thus, to relate the logarithmic derivative of $Z_\Sigma(s)$ with the determinant we need to understand the function $\phi_\Sigma(s)$ in terms of the heat kernel instead of the resolvent.

Lemma 7.2. *In the sense of distributions, we have that*

$$(\Delta_\Sigma + s)^{-1} = \int_0^\infty K_\Sigma(t)e^{-st}dt, \quad \textit{for } \operatorname{Re} s > 0.$$

Proof. Let $f \in \mathcal{C}_c^\infty(\Sigma)$ be a test function. Then by definition of the heat kernel, we have that

$$\partial_t K_\Sigma(t) f + \Delta_\Sigma K_\Sigma(t) f = 0, \quad K_\Sigma(0) f = f.$$

Using integration by part, this implies that

$$\begin{aligned} \Delta_\Sigma \left(\int_0^\infty K_\Sigma(t) f e^{-st} dt \right) &= \int_0^{-\infty} -\partial_t K_\Sigma(t) f e^{-st} dt \\ &= -K_\Sigma(t) f e^{-st}\big|_0^\infty - s \int_0^\infty K_\Sigma(t) f e^{-st} dt \\ &= f - s \int_0^\infty K_\Sigma(t) f e^{-st} dt, \end{aligned} \tag{7.15}$$

which shows that

$$(\Delta_\Sigma + s) \left(\int_0^\infty K_\Sigma(t) f e^{-st} dt \right) = f.$$

This means that

$$f \mapsto \int_0^\infty K_\Sigma(t) f e^{-st} dt$$

is a right inverse for $(\Delta_\Sigma + s)$. The same computation shows that it is a left inverse since

$$K_\Sigma(t) \Delta_\Sigma f = \Delta_\Sigma K_\Sigma(t) f$$

by uniqueness of the solution for the heat equation. □

From this lemma we conclude that

$$\frac{\varphi_\Sigma(s;z)}{2s-1} = \int_0^\infty \left(K_\Sigma(t,z,z) - K_{\mathbb{H}}(t;z,z) \right) e^{-s(s-1)t} dt. \tag{7.16}$$

Since the left-hand side is smooth, the right-hand side is smooth as well, which means that $K_\Sigma(t;z,z)$ and $K_{\mathbb{H}}(t;z,z)$ have the same term of order t^{-1} in their asymptotic expansions as $t \searrow 0$. Integrating (7.16) in z and taking the finite part, we get

$$\frac{\phi_\Sigma(s)}{2s-1} = {}^R\!\!\int_\Sigma \int_0^\infty \left(K_\Sigma(t;z,z) - K_{\mathbb{H}}(t;z,z) \right) e^{-ts(s-1)} dt dg_\Sigma(z). \tag{7.17}$$

The order of integration can be interchanged, since for $\mathrm{Re}(w) \gg 1$,

$$\begin{aligned} &\int_\Sigma \int_0^\infty x^w \left(K_\Sigma(t;z,z) - K_{\mathbb{H}}(t;z,z) \right) e^{-ts(s-1)} dt dg_\Sigma(z) = \\ &\qquad \int_0^\infty \int_\Sigma x^w \left(K_\Sigma(t;z,z) - K_{\mathbb{H}}(t;z,z) \right) e^{-ts(s-1)} dt dg_\Sigma(z). \end{aligned} \tag{7.18}$$

Hence, we get that

$$\frac{\phi_\Sigma(s)}{2s-1} = \int_0^\infty \left({}^R\mathrm{Tr}(K_\Sigma(t)) - {}^R\mathrm{Tr}_\Sigma(K_{\mathbb{H}}(t)) \right) e^{-ts(s-1)} dt \tag{7.19}$$

where

$$^R\operatorname{Tr}_\Sigma(K_{\mathbb{H}}(t)) := \mathrm{FP}\int_\Sigma K_{\mathbb{H}}(t;z,z)dg_\Sigma(z).$$

We recall that $K_{\mathbb{H}}(t)$ itself does not descend to $\Sigma\times\Sigma$, but its restriction to the diagonal in $\mathbb{H}\times\mathbb{H}$ does descend to the diagonal in $\Sigma\times\Sigma$. Indeed, it is well-known that because the hyperbolic metric on $\mathbb{H}$ is $\mathrm{SL}(2,\mathbb{R})$-invariant, $K_{\mathbb{H}}(t;z,z)$ is constant in z, say equal to $k_{\mathbb{H}}(t)dg_{\mathbb{H}}(z)$ for some function of t. Since the surfaces we are studying have finite area, we have

$$^R\operatorname{Tr}(K_{\mathbb{H}}(t)) = k_{\mathbb{H}}(t)\,\mathrm{area}(\Sigma).$$

Corollary 7.3. *The function $k_{\mathbb{H}}(t)$ has an asymptotic expansion given by*

$$k_{\mathbb{H}}(t)\sim k_{-1}t^{-1}+o(t^{-1}) \quad \text{as } t\searrow 0.$$

Therefore, if Σ is a Riemann surface of genus g with n cusps, then the regularized trace of its heat kernel has the asymptotic expansion

$$^R\operatorname{Tr}(K_\Sigma(t))\sim k_{-1}\,\mathrm{area}(\Sigma)t^{-1}+\mathcal{O}(t^{-\frac{1}{2}}\log t) \quad \text{as } t\searrow 0.$$

Proof. Consider the case where Σ has no cusp. Then it is well-known that $^R\operatorname{Tr}(K_\Sigma(t)) = \operatorname{Tr}(K_\Sigma(t))$ has asymptotic expansion of the form

$$\operatorname{Tr}(K_\Sigma(t))\sim \alpha t^{-1}+\beta+\mathcal{O}(t) \quad \text{as } t\searrow 0,$$

where α and β are some constants. By formula (7.19), $^R\operatorname{Tr}(K_{\mathbb{H}}(t))$ has the same term of order t^{-1} as $t\searrow 0$, hence we get that

$$k_{-1} := \frac{\alpha}{\mathrm{area}(\Sigma)} = \frac{1}{4\pi}$$

is such that

$$k_{\mathbb{H}}(t)\sim k_{-1}t^{-1}+o(t^{-1}) \quad \text{as } t\searrow 0.$$

When we consider a surface with cusps, $^R\operatorname{Tr}(K_\Sigma)$ will have the same term of order t^{-1} as $^R\operatorname{Tr}_\Sigma(K_{\mathbb{H}}(t))$ as $t\searrow 0$, hence

$$^R\operatorname{Tr}(K_\Sigma(t))\sim k_{-1}\,\mathrm{area}(\Sigma)t^{-1}+\mathcal{O}(t^{-\frac{1}{2}}\log t), \quad \text{as } t\searrow 0.$$

□

Consider the functional

$$D_\Sigma(s) := \det\left(\Delta_\Sigma + s(s-1)\right) = \exp\left(-\zeta'_{\Delta_\Sigma}(0;s(s-1))\right). \tag{7.20}$$

If we differentiate with respect s and use formula (7.9), we find that

$$\begin{aligned}\frac{1}{2s-1}\frac{D'_\Sigma(s)}{D_\Sigma(s)} = &\int_0^\infty \left(^R\operatorname{Tr}\left(e^{-t\Delta_\Sigma}\right)-a_{-1}t^{-1}\right)e^{-ts(s-1)}\,dt \\ &- a_{-1}\log\left(s(s-1)\right).\end{aligned} \tag{7.21}$$

Combining formula (7.21) and (7.19) and using corollary 7.3, we get that

$$\begin{aligned}(7.22)\quad \frac{1}{2s-1}\left(\frac{D'_\Sigma(s)}{D_\Sigma(s)}-\frac{Z'_\Sigma(s)}{Z_\Sigma(s)}\right) &= \\ \int_0^\infty \left({}^R\operatorname{Tr}(K_\Sigma(t)) - a_{-1}t^{-1} - {}^R\operatorname{Tr}(K_\Sigma(t)) + {}^R\operatorname{Tr}_\Sigma(K_{\mathbb{H}}(t))\right) & e^{-ts(s-1)}dt \\ - k_{-1}\operatorname{area}(\Sigma)\log\left(s(s-1)\right) &+ \frac{n}{2s-1}\phi_H(s) \\ = \int_0^\infty \left({}^R\operatorname{Tr}(K_{\mathbb{H}}(t)) - k_{-1}\operatorname{area}(\Sigma)t^{-1}\right) & e^{-ts(s-1)}dt \\ - k_{-1}\operatorname{area}(\Sigma)\log\left(s(s-1)\right) &+ \frac{n}{2s-1}\phi_H(s),\end{aligned}$$

so that

$$\begin{aligned}(7.23)\quad \frac{1}{2s-1}\left(\frac{D'_\Sigma(s)}{D_\Sigma(s)}-\frac{Z'_\Sigma(s)}{Z_\Sigma(s)}\right) &= \frac{n}{2s-1}\phi_H(s) \\ + \operatorname{area}(\Sigma)\left[\int_0^\infty \left(k_{\mathbb{H}}(t) - k_{-1}t^{-1}\right)e^{-ts(s-1)}dt - k_{-1}\log\left(s(s-1)\right)\right]&.\end{aligned}$$

In particular, for fixed g and n, we see that the right hand side does not depend on the Σ since the area is given by $-2\pi\chi(\Sigma)$ by the Gauss-Bonnet theorem, a quantity that only depends on g and n.

From (7.14), we see that

$$(7.24)\qquad \phi_H(s) = \frac{d}{ds}\left(-s\log 2 - \log\Gamma(s+\frac{1}{2}) + \frac{1}{2}\log(2s-1)\right).$$

Then, according to (7.23) and (7.4), there exists a constant C such that

$$(7.25)\qquad \frac{D_\Sigma(s)}{Z_\Sigma(s)} = C\left(e^E - s(s-1)\frac{\Gamma_2(s)^2}{\Gamma(s)}(2\pi)^s\right)^{-\chi(\Sigma)}\left(\frac{\sqrt{2s-1}}{2^s\Gamma(s+\frac{1}{2})}\right)^n$$

where $E := -\frac{1}{4} - \frac{1}{2}\log 2\pi + 2\zeta'(-1)$. As in [29], the constant C can be determined by the asymptotic expansion of the logarithm of both sides of (7.25) as s approaches infinity. For the left side, it is clear from (7.1) that $\log Z_\Sigma(s)$ has a trivial asymptotic expansion as $s\to+\infty$. Thus, from (7.9), we conclude that the asymptotic behavior of the logarithm of the left side is given by

$$(7.26)\qquad \begin{aligned}\log\left(\frac{D_\Sigma(s)}{Z_\Sigma(s)}\right) &= a_0\log s(s-1) + 2\sqrt{\pi}a_{-\frac{1}{2}}\sqrt{s(s-1)} \\ &- \widetilde{a}_{-\frac{1}{2}}\sqrt{s(s-1)}\left(\Gamma_{\log}(-\frac{1}{2}) + 2\sqrt{\pi}\log(s(s-1))\right) \\ &- a_{-1}s(s-1)\left(-1+\log(s(s-1))\right) + o(1)\end{aligned}$$

as $s \to +\infty$. On the other hand, if we set

$$Z_{\mathrm{cu}}(s) = \frac{\sqrt{2s-1}}{2^s\Gamma(s+\frac{1}{2})} = \frac{\sqrt{2}}{2^s\sqrt{s-\frac{1}{2}}\Gamma(s-\frac{1}{2})}, \tag{7.27}$$

we see using Stirling's formula that its logarithm has the following asymptotic behavior,

$$\log\left(Z_{\mathrm{cu}}(s)\right) = -\frac{1}{2}\log(2\pi) + (1-\log 2)\left(s-\frac{1}{2}\right) - \left(s-\frac{1}{2}\right)\log\left(s-\frac{1}{2}\right) + o(1) \tag{7.28}$$

as $s \to +\infty$. Since

$$\sqrt{s(s-1)} = s - \frac{1}{2} + o(1), \tag{7.29}$$

$$\sqrt{s(s-1)}\log\left(s(s-1)\right) = 2\left(s-\frac{1}{2}\right)\log\left(s-\frac{1}{2}\right) + o(1), \tag{7.30}$$

as $s \to +\infty$, we can rewrite (7.26) as

$$\begin{aligned}\log\left(\frac{D_\Sigma(s)}{Z_\Sigma(s)}\right) &= a_0\log s(s-1) + \left(2\sqrt{\pi}a_{-\frac{1}{2}} - \widetilde{a}_{-\frac{1}{2}}\Gamma_{\log}(-\frac{1}{2})\right)\left(s-\frac{1}{2}\right) \\ &\quad - 4\sqrt{\pi}\widetilde{a}_{-\frac{1}{2}}\left(s-\frac{1}{2}\right)\log\left(s-\frac{1}{2}\right) \\ &\quad - a_{-1}s(s-1)\left(-1+\log(s(s-1))\right) + o(1)\end{aligned} \tag{7.31}$$

as $s \to +\infty$. Now, from [29][1], we have also that

$$\begin{aligned}\log\left(e^{E-s(s-1)}\frac{\Gamma_2(s)^2}{\Gamma(s)}(2\pi)^s\right) &= \\ -\frac{1}{6}\log s(s-1) &+ \frac{1}{2}s(s-1) - \frac{s(s-1)}{2}\log s(s-1) + o(1)\end{aligned} \tag{7.32}$$

as $s \to +\infty$. This asymptotic behavior only involves terms of the form $\log\left(s(s-1)\right)$ and $s(s-1)\log\left(s(s-1)\right)$. Thus, in (7.31), the terms involving $a_{-\frac{1}{2}}$ and $\widetilde{a}_{-\frac{1}{2}}$ counterbalance the asymptotic behavior of (7.28) while the terms involving a_{-1} and a_0 counterbalance the asymptotic behavior of (7.32). Comparing (7.31) with (7.32), we find

$$a_{-1} = g - 1 = -\frac{\chi(\Sigma)}{2}, \quad a_0 = \frac{\chi(\Sigma)}{6}. \tag{7.33}$$

Comparing (7.31) with (7.28), we also get

$$\widetilde{a}_{-\frac{1}{2}} = \frac{n}{4\sqrt{\pi}}, \quad a_{-\frac{1}{2}} = \frac{n}{2\sqrt{\pi}}\left(1 - \log 2 + \frac{\Gamma_{\log}(-\frac{1}{2})}{4\sqrt{\pi}}\right). \tag{7.34}$$

[1]In (2.19) of [29], the coefficient of $\log s(s-1)$ is $\frac{1}{24}$, but it is supposed to be $\frac{1}{6}$

Now, recall that in [29], the constant E is chosen so that

$$\log\left(e^{E-s(s-1)}\frac{\Gamma_2(s)^2}{\Gamma(s)}(2\pi)^s\right) = \\ -a_{-1}s(s-1)\left(-1+\log\left(s(s-1)\right)\right)+a_0\log\left(s(s-1)\right). \tag{7.35}$$

This means the constant C has to be chosen to compensate the constant term of (7.28), that is,

$$\log C = -\frac{n}{2}\log 2\pi \quad \Longrightarrow \quad C=(2\pi)^{-\frac{n}{2}}. \tag{7.36}$$

This gives the following result.

Theorem 2. *For a Riemann surface of genus g with n cusps satisfying $2g-2+n>0$ and equipped with the hyperbolic metric, we have*

$$\det\left(\Delta_\Sigma+s(s-1)\right)=Z_\Sigma(s)\frac{\left(e^{E-s(s-1)}\frac{\Gamma_2(s)^2}{\Gamma(s)}(2\pi)^s\right)^{-\chi(\Sigma)}}{\left(2^s\sqrt{\pi(s-\frac{1}{2})}\Gamma(s-\frac{1}{2})\right)^n}.$$

As a consequence, we see that the ratio

$$\frac{\det\left(\Delta_\Sigma+s(s-1)\right)}{Z_\Sigma(s)}$$

is a meromorphic function in s which only depends on the genus g and the number of cusps n. This means that, up to a multiplicative constant depending only on g and n, the determinant of Δ_Σ is given by $Z'_\Sigma(1)$. The formula of theorem 2 can also be expressed in terms of the $\overline{\partial}$-Laplacian

$$\Delta_{\overline{\partial}}=\frac{1}{2}\Delta_\Sigma.$$

For the heat kernel of the $\overline{\partial}$-Lapacian, we have the following short time asymptotic expansion,

$$\begin{aligned} {}^R\operatorname{Tr}\left(e^{-t\Delta_{\overline{\partial}}}\right) &= {}^R\operatorname{Tr}\left(e^{-\frac{t}{2}\Delta_\Sigma}\right) \\ &= \frac{2a_{-1}}{t}+\frac{\sqrt{2}\widetilde{a}_{-\frac{1}{2}}}{\sqrt{t}}\log t+\frac{\sqrt{2}a_{-\frac{1}{2}}-\sqrt{2}\widetilde{a}_{-\frac{1}{2}}\log 2}{\sqrt{t}}+a_0+\mathcal{O}(\sqrt{t}) \end{aligned} \tag{7.37}$$

as $t\to 0^+$, where $a_{-1},\widetilde{a}_{-\frac{1}{2}},a_{-\frac{1}{2}},a_0$ are the coefficients in (7.5). From formula (7.7), we conclude that

$$\zeta_{\Delta_{\overline{\partial}}}\left(z;\frac{s(s-1)}{2}\right)=2^z\zeta_{\Delta_\Sigma}\left(z;s(s-1)\right).$$

Hence,

$$\zeta'_{\Delta_{\overline{\partial}}}\left(0;\frac{s(s-1)}{2}\right)=(\log 2)\,\zeta_{\Delta_\Sigma}\left(0;s(s-1)\right)+\zeta'_\Sigma\left(0;s(s-1)\right),$$

which means that

$$\det\left(\Delta_{\overline{\partial}} + \frac{s(s-1)}{2}\right) = 2^{-\zeta_{\Delta_\Sigma}(0;s(s-1))} \det\left(\Delta_\Sigma + s(s-1)\right).$$

Now, $\zeta_{\Delta_\Sigma}(0; s(s-1))$ can be computed explicitly from (7.7) and (7.33),

$$\begin{aligned} \zeta_{\Delta_\Sigma}(0; s(s-1)) &= a_0 - a_{-1}s(s-1) \\ &= \chi(\Sigma)\left(\frac{1}{6} + \frac{s(s-1)}{2}\right). \end{aligned} \tag{7.38}$$

From theorem 2, we get the following formula.

Corollary 7.4. *For a Riemann surface Σ of genus g with n cusps satisfying $2g-2+n>0$ and equipped with the hyperbolic metric, we have*

$$\det\left(\Delta_{\overline{\partial}} + \frac{s(s-1)}{2}\right) = Z_\Sigma(s)\frac{\left(2^{\frac{1}{6}+\frac{s(s-1)}{2}} e^{E-s(s-1)} \frac{\Gamma_2(s)^2}{\Gamma(s)}(2\pi)^s\right)^{-\chi(\Sigma)}}{\left(2^s\sqrt{\pi(s-\frac{1}{2})}\Gamma(s-\frac{1}{2})\right)^n}.$$

7.3. The determinant of Δ_ℓ for $\ell \geq 1$.

As indicated in [29], it is possible to express the determinant of Δ_ℓ in terms of Selberg Zeta function by using corollary 7.4. This is because the spectrum of Δ_ℓ is essentially given by a shifted version of the spectrum of $\Delta_0 = \overline{\partial}^*\overline{\partial}$.

Recall first that these various Laplacians are related by the recurrence relation (see[2] for instance (1.3) in [30])

$$\Delta_\ell \overline{\partial}_\ell^* u = \overline{\partial}_\ell^* u \left(\Delta_{\ell-1} + \ell - 1\right) \tag{7.39}$$

where $u := \frac{1}{y^2}$ is seen as a section of $\Lambda^{1,0}_\Sigma \otimes \Lambda^{0,1}_\Sigma$ on Σ. Taking the formal adjoint of (7.39), we get

$$u^*\overline{\partial}_\ell \Delta_\ell = \left(\Delta_{\ell-1} + \ell - 1\right) u^*\overline{\partial}_\ell \tag{7.40}$$

where u^* is the conjugate $\overline{u}$ of u seen as a section of $(\Lambda^{1,0}_\Sigma)^{-1} \otimes (\Lambda^{0,1}_\Sigma)^{-1}$.

Since the operator $\overline{\partial}_\ell$ is Fredholm, it has a well-defined parametrix $\overline{\partial}_\ell^{-1}$: $(\ker \overline{\partial}_\ell^*)^\perp \to (\ker \overline{\partial}_\ell)^\perp$. Applying this parametrix to both sides of (7.40), we get

$$\Delta_\ell = \left(u^*\overline{\partial}_\ell\right)^{-1}\left(\Delta_{\ell-1} + \ell - 1\right)\left(u^*\overline{\partial}_\ell\right). \tag{7.41}$$

In the compact case, this directly implies that

$$\det{}'(\Delta_\ell) = \det(\Delta_{\ell-1} + \ell - 1) \tag{7.42}$$

for $\ell \geq 2$ since $\overline{\partial}_\ell$ is surjective in that case. When $\ell = 1$, the operator $\overline{\partial}_1$ is not surjective, but the cokernel of $u^*\overline{\partial}_\ell$,

$$\ker \overline{\partial}_1^* u = \ker \overline{\partial}_0 = \ker \Delta_0$$

[2]We have $\ell - 1$ instead of $\frac{\ell-1}{2}$ in [30] since we use the convention $|dz|^2 = 2$.

is precisely the kernel of Δ_0, so that we have in that case

$$\det{}'(\Delta_1) = \det{}'(\Delta_0).$$

Using (7.41) once more and (3.25), we have on the other hand that for $k > 0$,

$$\det(\Delta_{\ell-1} + k) = \begin{cases} (k)^{g-1} \det(\Delta_0 + k)\,, & \ell = 2; \\ (k)^{(2\ell-1)(g-1)} \det(\Delta_{\ell-2} + k + \ell - 2)\,, & \ell \geq 3. \end{cases} \tag{7.43}$$

Applying this recursively, we get

$$\det{}'(\Delta_\ell) = \begin{cases} \det{}'(\Delta_0), & \ell = 1; \\ \det(\Delta_0 + 1)\,, & \ell = 2; \\ \delta_{\ell,g} \det(\Delta_0 + \ell(\ell - 1))\,, & \ell \geq 3. \end{cases} \tag{7.44}$$

where $\delta_{\ell,g}$ is a number depending only on ℓ and g. In the non-compact case, one has to be more careful since the regularized trace does not necessarily vanish on a commutator. Taking this into account, the analog of (7.41) in the non-compact case is

$$\det{}'(\Delta_\ell) = D_{\ell,n} \det(\Delta_{\ell-1} + \ell - 1) \tag{7.45}$$

with

(7.46)

$$-\log(D_{\ell,n}) = \left(\frac{d}{dz} \frac{1}{\Gamma(z)} \int_0^\infty t^{zR}\, \mathrm{Tr}\left([(u^*\overline{\partial}_\ell)^{-1} e^{-t\Delta_{\ell-1}}, u^*\overline{\partial}_\ell] \right) e^{-\frac{t(\ell-1)}{2}} \frac{dt}{t} \right)_{z=0}$$

regularizing as in (7.6). Although the term $D_{\ell,n}$ might be hard to compute, what is clear is that it only depends on ℓ and the number n of cusps. This is because the regularized trace of a commutator $[A, B]$ 'localizes' near the boundary in the sense that it only depends on the Taylor expansion of the integral kernels at the boundary of the diagonal. Recall that to construct the heat kernel (see [31, 2]) we start with a 'parametrix' for the heat equation which solves a model equation at the cusp. The solution of this model equation is then used iteratively to construct the Taylor expansion of the heat kernel as we approach the cusp, before finally solving away the remaining error in the interior. The upshot is that, since all cusps have isometric neighborhoods, the term $D_{\ell,n}$ only depends on ℓ and n as required.

Thus using recursively (7.45) and applying corollary 7.4, we get the following.

Corollary 7.5. *For a Riemann surface Σ of genus $g \geq 2$ with n cusp, we have*

$$\det{}'(\Delta_\ell) = \begin{cases} \alpha_{\ell,g,n} Z_\Sigma(\ell), & \ell \geq 2; \\ \alpha_{\ell,g,n} Z'_\Sigma(1), & \ell = 0, 1; \end{cases}$$

where each constant $\alpha_{\ell,g,n} > 0$ only depends on ℓ, g and n. A similar statement holds of the determinant of $D_\ell^- D_\ell^+ = 2\Delta_\ell$.

8. The curvature of the Quillen connection

Recall that the determinant bundle of the family of operator $\overline{\partial}_\ell$ is by definition

$$\lambda_\ell := \det \operatorname{ind} \overline{\partial}_\ell = \Lambda^{\max} \ker \overline{\partial}_\ell \otimes \left(\Lambda^{\max} \operatorname{coker} \overline{\partial}_\ell\right)^{-1} \tag{8.1}$$

where $\ell \in \mathbb{Z}$ and $\Lambda^{\max}$ denotes the maximal exterior power of a vector space. The definition is particularly simple in this case because $\ker \overline{\partial}_l$ is a vector bundle over $T_{g,n}$. The L^2-norm on $\ker \overline{\partial}_\ell$ defines a canonical metric on λ_ℓ, the L^2-metric, denoted $\|\cdot\|$. An alternative metric which is more interesting geometrically is the **Quillen metric**,

$$\|\cdot\|_Q := \left(\det D_\ell^- D_\ell^+\right)^{-\frac{1}{2}} \|\cdot\|. \tag{8.2}$$

Following the discussion of § 9.7 in [5], we will associate to $\|\cdot\|_Q$ a compatible connection called the **Quillen connection**. In order to do that, consider over $\mathcal{T}_{g,n}$ the $\mathbb{Z}_2$-graded bundle

$$\mathcal{E}_\ell = \mathcal{E}_\ell^+ \oplus \mathcal{E}_\ell^-, \quad \mathcal{E}_\ell^+ := \Lambda^{\ell,0}\left(\mathcal{T}_{g,n}/T_{g,n}\right), \quad \mathcal{E}_\ell^- := \Lambda^{\ell,1}\left(\mathcal{T}_{g,n}/T_{g,n}\right). \tag{8.3}$$

Let also $\pi_*\mathcal{E}_\ell \to T_{g,n}$ be the Fréchet bundle whose fiber at $[\Sigma] \in T_{g,n}$ is

$$\pi_*\mathcal{E}_{\ell,[\Sigma]} := \dot{\mathcal{C}}^\infty\left(\Sigma; \mathcal{E}_l|_\Sigma \otimes |\Lambda_\Sigma|^{\frac{1}{2}}\right), \tag{8.4}$$

where $|\Lambda_\Sigma|$ is the density bundle on Σ and $\dot{\mathcal{C}}^\infty\left(\Sigma; \mathcal{E}_l|_\Sigma \otimes |\Lambda_\Sigma|^{\frac{1}{2}}\right)$ is the space of smooth sections of $\mathcal{E}_l|_\Sigma \otimes |\Lambda_\Sigma|^{\frac{1}{2}}$ with rapid decay at infinity. The family of Dirac type operators

$$D_\ell := \sqrt{2}\left(\overline{\partial}_\ell + \overline{\partial}_\ell^*\right), \quad D_\ell^+ = \sqrt{2}\overline{\partial}_\ell, \quad D_\ell^- = \sqrt{2}\overline{\partial}_\ell^*, \tag{8.5}$$

acts from $\pi_*\mathcal{E}_\ell$ to $\pi_*\mathcal{E}_\ell$. One of the reasons that motivates the introduction of the fibre density bundle in the definition of $\pi_*\mathcal{E}$ is that in this way the canonical connection on $\pi : \mathcal{T}_{g,n} \to T_{g,n}$ induces a connection on $\pi_*\mathcal{E}_\ell$, denoted $\nabla^{\pi_*\mathcal{E}_\ell}$, which is automatically compatible with the metric of $\pi_*\mathcal{E}_\ell$ (cf. proposition 9.13 in [5]). Notice also that the density bundle $|\Lambda_\Sigma|$ is canonically trivialized by the section $|dg_\Sigma|$ so that D_ℓ acts on $\pi_*\mathcal{E}_\ell$ in a natural way. To the family of Dirac type operators D_ℓ, we can associate a superconnection

$$\mathbb{A}_\ell := D_\ell + \nabla^{\pi_*\mathcal{E}_\ell} \tag{8.6}$$

and its rescaled version

$$\mathbb{A}_\ell^s := s^{\frac{1}{2}} D_\ell + \nabla^{\pi_*\mathcal{E}_\ell}. \tag{8.7}$$

For $s \in \mathbb{R}^+$, we can define two differential forms $\alpha_\ell^\pm \in \mathcal{A}(T_{g,n})(s)$,

$$\begin{aligned}\alpha_\ell^\pm(s) &:= {}^R\operatorname{Tr}_{\pi_*\mathcal{E}_\ell^\pm}\left(\frac{\partial \mathbb{A}_\ell^s}{\partial s} e^{-(\mathbb{A}_\ell^s)^2}\right) \\ &= \frac{1}{2s^{\frac{1}{2}}}{}^R\operatorname{Tr}_{\pi_*\mathcal{E}_\ell^\pm}\left(D_\ell e^{-(\mathbb{A}_\ell^s)^2}\right),\end{aligned} \tag{8.8}$$

by taking the trace with respect to $\mathcal{E}_\ell^+$ and $\mathcal{E}_\ell^-$ respectively. The 0-form component of $(\mathbb{A}_\ell^2)^s$ is sD_ℓ^2, while its 1-form component is $s^{\frac{1}{2}}[\nabla^{\pi_*\mathcal{E}_l}, D_\ell]$. On the other hand, the 1-form component of $e^{-(\mathbb{A}_\ell^s)^2}$ is given by

$$\begin{aligned}\left(e^{-(\mathbb{A}_\ell^s)^2}\right)_{[1]} &= (-s)\int_0^1 e^{-(1-\sigma)sD_\ell^2} s^{-\frac{1}{2}}[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell] e^{-\sigma s D_\ell^2} d\sigma \\ &= -s^{\frac{1}{2}}\int_0^1 e^{-(1-\sigma)sD_\ell^2}[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell] e^{-\sigma s D_\ell^2} d\sigma.\end{aligned} \tag{8.9}$$

The following observation will turn out to be very useful.

Lemma 8.1. *The Schwartz kernel of $[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^\pm]$ vanishes to all order at the front face. In particular, for $P \in \Psi^{-\infty}(\mathcal{T}_{g,n}/T_{g,n}; \mathcal{E}_\ell)$,*

$$ {}^R\operatorname{STr}\left([[\nabla^{\pi_*\mathcal{E}_\ell}, D_l^\pm], P]\right) = 0$$

Proof. Let $[\Sigma] \in T_{g,n}$ be given. If $\mu \in \Omega^{-1,1}(\Sigma)$ is a harmonic Beltrami differential, let $f^\mu : \mathbb{H} \to \mathbb{H}$ be the unique diffeomorphism satisfying the Beltrami equation

$$\frac{\partial f^\mu}{\partial \overline{z}} = \mu \frac{\partial f^\mu}{\partial z}$$

and fixing the points $0, 1, \infty$. In particular, since μ is a cusp form, it decreases rapidly as $z \to \infty$. This means that f^μ is asymptotically holomorphic as $z \to \infty$. From the definition of the canonical connection on $\pi : \mathcal{T}_{g,n} \to T_{g,n}$, this means that the Schwartz kernel $[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^+](z, z')$ decreases quickly as z and z' approaches a cusp in Σ. For $D_\ell^- = \sqrt{2}\overline{\partial}_\ell^*$, the same is true, but since $\overline{\partial}_\ell^* = -u^{\ell-1}\partial u^{-\ell}$, we also need to use the fact that $u = \frac{1}{y^2}$ is parallel with respect to the canonical connection on $\pi : \mathcal{T}_{g,n} \to T_{g,n}$. Now, we know that ${}^R\operatorname{STr}\left([[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^\pm], P]\right)$ depends linearly on the asymptotic expansions of $[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^\pm]$ and P at the corner of $\Sigma \times \Sigma$. The asymptoptic expansion of $[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^\pm]$ being trivial, the result follows.

Alternatively, the result also follows directly from the explicit formulas (5.15). □

With this lemma, the discussion of § 9.7 in [5] applies almost directly to our context.

Lemma 8.2. *The one form component of the differential forms $\alpha_\ell^+(s)$ satisfies*

$$\overline{\alpha_\ell^+(s)_{[1]}} = \alpha_\ell^-(s)_{[1]}$$

and has an asymptotic expansion of the form

$$\alpha_\ell^+(s)_{[1]} \sim \sum_{-N}^{\infty} s^{\frac{k}{2}}(a_k + b_k \log s)$$

as $s \to 0^+$.

Proof. The asymptotic expansion as $s \to 0^+$ follows from the construction of the heat kernel by Vaillant [31], its generalization in [2] and an application of the pushforward theorem for manifolds with corners. From (8.9), we have that

$$\begin{aligned} \alpha_\ell^+(s)_{[1]} &= -\frac{1}{2}{}^R\operatorname{Tr}_{\pi_*\mathcal{E}_\ell^+}\left(D_\ell \int_0^1 e^{-(1-\sigma)sD_\ell^2}[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell]e^{-\sigma s D_\ell^2}d\sigma\right) \\ &= -\frac{1}{2}{}^R\operatorname{STr}_{\pi_*\mathcal{E}_\ell}\left(D_\ell^- \int_0^1 e^{-(1-\sigma)sD_\ell^2}[\nabla^{\pi_*\mathcal{E}_l}, D_\ell^+]e^{-\sigma s D_\ell^2}d\sigma\right). \end{aligned} \tag{8.10}$$

Taking the complex conjugate and using the fact that $(D_\ell^+)^* = D_\ell^-$ and that $\nabla^{\pi_*\mathcal{E}_\ell}$ is a unitary connection, we have that

(8.11)

$$\begin{aligned} \overline{\alpha_\ell^+(s)_{[1]}} &= -\frac{1}{2}{}^R\operatorname{Tr}_{\pi_*\mathcal{E}_\ell^+}\left(\int_0^1 e^{-\sigma sD_\ell^2}[D_\ell^-, \nabla^{\pi_*\mathcal{E}_\ell}]e^{-(1-\sigma)sD_\ell^2}D_\ell^+ d\sigma\right) \\ &= \frac{1}{2}{}^R\operatorname{STr}_{\pi_*\mathcal{E}_\ell}\left(\int_0^1 e^{-(1-\sigma)sD_\ell^2}[D_\ell^-, \nabla^{\pi_*\mathcal{E}_\ell}]e^{-\sigma sD_\ell^2}D_\ell^+ d\sigma\right) \\ &= \frac{1}{2}{}^R\operatorname{STr}_{\pi_*\mathcal{E}_\ell}\left(D_\ell^+\int_0^1 e^{-(1-\sigma)sD_\ell^2}[D_\ell^-, \nabla^{\pi_*\mathcal{E}_\ell}]e^{-\sigma sD_\ell^2} d\sigma\right) \\ &\quad + \frac{1}{2}{}^R\operatorname{STr}_{\pi_*\mathcal{E}_\ell}\left(\left[\int_0^1 e^{-(1-\sigma)sD_\ell^2}[D_\ell^-, \nabla^{\pi_*\mathcal{E}_\ell}]e^{-\sigma sD_\ell^2} d\sigma, D_\ell^+\right]\right) \\ &= -\frac{1}{2}{}^R\operatorname{Tr}_{\pi_*\mathcal{E}_\ell^-}\left(D_\ell^+\int_0^1 e^{-(1-\sigma)sD_\ell^2}[D_\ell^-, \nabla^{\pi_*\mathcal{E}_\ell}]e^{-\sigma sD_\ell^2} d\sigma\right) + 0 \\ &= \alpha_\ell^-(s)_{[1]}, \end{aligned}$$

where Lemma 8.1 was used in the line before the last one. □

We would like to consider the one-forms

$$\beta_\ell^\pm(z) := 2\int_0^\infty t^z \alpha_\ell^\pm(t)_{[1]}dt. \tag{8.12}$$

By Lemma 8.2, this integral is holomorphic for $\operatorname{Re} z >> 0$ and admits a meromorphic extension to the whole complex plane. Thus, in this sense, $\beta_\ell^+(z)$ and $\beta_\ell^-(z)$ are well-defined meromorphic families of one-forms. We are interested in their finite part at $z=0$. More precisely, we would like to consider the one-forms

$$\beta_\ell^\pm := \frac{d}{dz}\frac{1}{\Gamma(z)}\beta_\ell^\pm(z)\bigg|_{z=0} \tag{8.13}$$

where the evaluation at zero means that we take the finite part at $z=0$. More generally, we will use the notation

$$\overline{\int_0^\infty} \gamma dt := \left(\frac{d}{dz} \frac{1}{\Gamma(z)} \int_0^\infty t^z \gamma(t) dt \right)_{z=0}$$

whenever the integral $\int_0^\infty t^{z-1}\gamma(t)dt$ varies meromorphically in z. Thus, in this notation,

$$\beta_l^{\pm} = 2\overline{\int_0^\infty} \alpha_\ell^{\pm}(t)_{[1]} dt.$$

Lemma 8.3. *Seen as a function on $T_{g,n}$, the differential of $\zeta'(0; D_\ell^- D_\ell^+)$ is given by $d\zeta'(0; D_\ell^- D_\ell^+) = -\left(\beta_\ell^+ + \beta_\ell^-\right)$.*

Proof. Using Duhamel's formula, we have that $d\zeta'(0; D_\ell^- D_\ell^+)$ is given by

$$\begin{aligned}(8.14)\quad &\overline{\int_0^\infty}{}^R \operatorname{Tr}_{\pi_*\mathcal{E}_\ell^+} \left(-\frac{1}{t} \int_0^t e^{-(t-s)D_\ell^- D_\ell^+} [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^- D_\ell^+] e^{-sD_\ell^- D_\ell^+} ds \right) dt = \\ &\overline{\int_0^\infty}{}^R \operatorname{Tr}_{\pi_*\mathcal{E}_\ell^+} \left(-\int_0^1 e^{-(1-s)tD_\ell^- D_\ell^+} [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^- D_\ell^+] e^{-stD_\ell^- D_\ell^+} ds \right) dt.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\beta_\ell^+ &= 2\overline{\int_0^\infty} \alpha_\ell^+(t)_{[1]} dt \\ (8.15)\qquad &= -\overline{\int_0^\infty}{}^R \operatorname{Tr}_{\pi_*\mathcal{E}_\ell^+} \left(D_\ell^- \int_0^1 e^{-(1-s)tD_\ell^2} [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^+] e^{-stD_\ell^2} ds \right) dt \\ &= -\overline{\int_0^\infty}{}^R \operatorname{Tr}_{\pi_*\mathcal{E}_\ell^+} \left(\int_0^1 e^{-(1-s)tD_\ell^2} D_\ell^- [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^+] e^{-stD_\ell^2} ds \right) dt,\end{aligned}$$

while
(8.16)

$$\begin{aligned}\beta_\ell^- &= -\overline{\int_0^\infty}{}^R \operatorname{Tr}_{\pi_*\mathcal{E}_\ell^-} \left(D_\ell^+ \int_0^1 e^{-(1-s)tD_\ell^2} [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^-] e^{-stD_\ell^2} ds \right) dt \\ &= \overline{\int_0^\infty}{}^R \operatorname{STr}_{\pi_*\mathcal{E}_\ell} \left(D_\ell^+ \int_0^1 e^{-(1-s)tD_\ell^2} [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^-] e^{-stD_\ell^2} ds \right) dt \\ &= \overline{\int_0^\infty}{}^R \operatorname{STr}_{\pi_*\mathcal{E}_\ell} \left(\int_0^1 e^{-(1-s)tD_\ell^2} [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^-] D_\ell^+ e^{-stD_\ell^2} ds \right) dt \\ &\quad + \overline{\int_0^\infty}{}^R \operatorname{STr}_{\pi_*\mathcal{E}_\ell} \left(\left[D_\ell^+, \int_0^1 e^{-(1-s)tD_\ell^2} [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^-] e^{-stD_\ell^2} ds \right] \right) dt \\ &= \overline{\int_0^\infty}{}^R \operatorname{STr}_{\pi_*\mathcal{E}_\ell} \left(\int_0^1 e^{-(1-s)tD_\ell^2} [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^-] D_\ell^+ e^{-stD_\ell^2} ds \right) dt + 0,\end{aligned}$$

using Lemma 8.1 in the last step. The result then follows by combining (8.14), (8.15) and (8.16) and using the formula

$$[\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^- D_\ell^+] = [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^-] D_\ell^+ - D_\ell^- [\nabla^{\pi_*\mathcal{E}_\ell}, D_\ell^+].$$

□

If $P_0 : \pi_* \mathcal{E}_\ell^+ \to \ker \overline{\partial}_\ell$ denotes the orthogonal projection onto the kernel of $\overline{\partial}_\ell$, then the connection

$$\nabla^{\ker \overline{\partial}_\ell} = P_0 \nabla^{\pi_* \mathcal{E}_\ell^+} P_0 \tag{8.17}$$

is compatible with the L^2-metric. It is holomorphic, so that $\nabla^{\ker \overline{\partial}_\ell}$ is the Chern connection of $\ker \overline{\partial}_\ell$ with respect to the L^2-metric $\| \cdot \|$. It defines a connection on $\det \overline{\partial}_\ell$, $\nabla^{\det \overline{\partial}_\ell}$, which is the Chern connection of $\det \overline{\partial}_\ell$ with respect to the L^2-metric. We define the **Quillen connection** on $\det \overline{\partial}_\ell$ to be the connection given by

$$\nabla^{Q_\ell} := \nabla^{\det \overline{\partial}_\ell} + \beta_\ell^+. \tag{8.18}$$

Proposition 8.4. *The Quillen connection is the Chern connection of* $\det \overline{\partial}_\ell$ *with respect to the Quillen metric* $\| \cdot \|_{Q_\ell}$.

Proof. We need to check that ∇^{Q_ℓ} is holomorphic and is compatible with the Quillen metric. To see that it is holomorphic, it suffices to check that β_ℓ^+ is a $(1,0)$-form. Since $D_\ell^+ = \sqrt{2} \overline{\partial}_\ell$ is a family of operators that varies holomorphically on $T_{g,n}$, the form

$$[\nabla^{\pi_* \mathcal{E}_\ell}, D_\ell^+]$$

has to be a $(1,0)$-form (cf. with (5.15)). Directly from the definition of β_ℓ^+, we thus see it has to be a $(1,0)$-form.

To see that ∇^{Q_ℓ} is compatible with the Quillen metric, notice that in general, a connection which is compatible with the Quillen metric is of the form

$$\nabla^{\det \overline{\partial}_l} - \frac{1}{2} d\zeta' \left(0; D_\ell^- D_\ell^+\right) + \omega \tag{8.19}$$

where ω is any imaginary one-form. The result then follows by noticing that, taking $\omega = \frac{\beta_\ell^+ - \overline{\beta_\ell^+}}{2}$ and using lemma 8.2 and lemma 8.3, we get the Quillen connection. □

We can now compute the curvature of the Quillen connection.

Theorem 3. *The curvature of the Quillen connection is given by*

$$\frac{\sqrt{-1}}{2\pi} (\nabla^{Q_\ell})^2 = \left(\int_{\mathcal{T}_{g,n}/T_{g,n}} \mathrm{Ch} \left(T^{-\ell} (\mathcal{T}_{g,n}/T_{g,n}) \right) \cdot \mathrm{Td} \left(T \left(\mathcal{T}_{g,n}/T_{g,n} \right) \right) \right)_{[2]} - \sum_{i=1}^{n} \frac{e_i}{12}$$

Proof. With respect to the connection $\nabla^{\det \overline{\partial}_\ell}$, we have

$$\frac{\sqrt{-1}}{2\pi} \left(\nabla^{\det \overline{\partial}_l} \right)^2 = \mathrm{Ch} \left(\nabla^{\ker \overline{\partial}_l} \right)_{[2]} . \tag{8.20}$$

But by definition, since $\nabla^{\pi_*\mathcal{E}_\ell} = \mathbb{A}_{[1]}$ for $\mathbb{A}$ the Bismut superconnection, (cf. Proposition 10.16 in [5]), we have by (8.17) that $\nabla^{\ker \overline{\partial}_l}$ is the connection used in Theorem 1. Thus, $\mathrm{Ch}\left(\nabla^{\ker\overline{\partial}_\ell}\right)_{[2]}$ is given by formula (6.16), so that

(8.21)
$$\frac{\sqrt{-1}}{2\pi}(\nabla^{\det\overline{\partial}_\ell})^2 = \left(\int_{\mathcal{T}_{g,n}/T_{g,n}} \mathrm{Ch}\left(T^{-\ell}(\mathcal{T}_{g,n}/T_{g,n})\right)\cdot \mathrm{Td}\left(T\left(\mathcal{T}_{g,n}/T_{g,n}\right)\right)\right)_{[2]} - \sum_{i=1}^n \frac{e_i}{12} - \frac{1}{2\pi\sqrt{-1}} d\int_0^\infty {}^R\,\mathrm{STr}\left(\frac{\partial\mathbb{A}_t}{\partial t}e^{-\mathbb{A}_t^2}\right)_{[1]} dt.$$

On the other hand, from the definition of the Quillen connection, we have

$$\left(\nabla^{Q_\ell}\right)^2 = \left(\nabla^{\det\overline{\partial}_l}\right)^2 + d\beta_\ell^+. \tag{8.22}$$

From lemma 8.3, $d(\beta_\ell^+ + \beta_\ell^-) = 0$, hence

$$\left(\nabla^{Q_\ell}\right)^2 = \left(\nabla^{\det\overline{\partial}_l}\right)^2 + \frac{1}{2}d\left(\beta_\ell^+ - \beta_\ell^-\right). \tag{8.23}$$

But using the fact the Bismut superconnection $\mathbb{A}$ is given by $D_\ell + \nabla^{\pi_*\mathcal{E}}$ up to terms of degree 2, we have

(8.24)
$$\frac{1}{2}\left(\beta_\ell^+ - \beta_\ell^-\right) = \overline{\int_0^\infty}\left(\alpha_\ell^+(t)_{[1]} - \alpha_\ell^-(t)_{[1]}\right)dt = \overline{\int_0^\infty} {}^R\,\mathrm{STr}\left(\frac{\partial\mathbb{A}_t}{\partial t}e^{-\mathbb{A}_t^2}\right)_{[1]} dt = \int_0^\infty {}^R\,\mathrm{STr}\left(\frac{\partial\mathbb{A}_t}{\partial t}e^{-\mathbb{A}_t^2}\right)_{[1]} dt.$$

In the last step, we have used the fact ${}^R\,\mathrm{STr}\left(\frac{\partial\mathbb{A}_t}{\partial t}e^{-\mathbb{A}_t^2}\right)$ is integrable in t, so that there is no need to regularize. Combining (8.21), (8.23) and (8.24), the result follows. □

We should compare our result with the local index formula of Takhtajan and Zograf [30]

$$(\nabla^{Q_\ell})^2 = \frac{6\ell^2 - 6\ell + 1}{12\pi^2}\omega_{WP} - \frac{1}{9}\omega_{\mathrm{TZ}}, \tag{8.25}$$

where ω_{WP} is the Weil-Peterson Kähler form on $T_{g,n}$ and ω_{TZ} is the Kähler form on $T_{g,n}$ defined by Takhtajan and Zograf in terms of the cusp ends of the fibres of $p : \mathcal{T}_{g,n} \to T_{g,n}$. A well-known result of Wolpert [34] (see also

p. 424 in [30]) shows that

$$\left(\int_{\mathcal{T}_{g,n}/T_{g,n}} \mathrm{Ch}\left(T^{-\ell}(\mathcal{T}_{g,n}/T_{g,n})\right)\cdot \mathrm{Td}\left(T\left(\mathcal{T}_{g,n}/T_{g,n}\right)\right)\right)_{[2]} = \frac{6\ell^2-6\ell+1}{12\pi^2}\omega_{WP}. \tag{8.26}$$

Thus, comparing Theorem 3 with (8.25), we get the following relation.

Corollary 8.5 (Weng [32], Wolpert [35]). *For $\ell \geq 0$ and $n > 0$, we have*

$$\widehat{\eta}(D_\ell^V)_{[2]} = \sum_{i=1}^n \frac{e_i}{12} = \frac{1}{9}\omega_{\mathrm{TZ}}.$$

The fact the Takhtajan-Zograf Kähler form is a rational multiple of the curvature of a Hermitian line bundle was first obtained by Weng [32] using Arakelov theory. This was later improved and finalized by Wolpert [35], who obtained more generally that $e_i = \frac{4}{3}\omega_{\mathrm{TZ},i}$ ($\omega_{\mathrm{TZ},i}$ is defined in (8.31) below) via a natural intrinsic way to define metrics on the line bundles $\mathcal{L}_i$.

For completeness, let us recall how the Takhtajan-Zograf Kähler form ω_{TZ} is defined. Given a fibre Σ of $p: \mathcal{T}_{g,n} \to T_{g,n}$, identify it with a quotient of the upper half-plane, $\Sigma \cong \Gamma \setminus \mathbb{H}$ where Γ is the corresponding Fuchsian group of type (g,n). Let $\Gamma_1, \ldots, \Gamma_n$ be the list of non-conjugate parabolic subgroup of Γ as in (5.2) so that

$$\sigma_i^{-1}\Gamma_i\sigma_i = \Gamma_\infty$$

for $i \in \{1,\ldots,n\}$. The Eisenstein-Mass series $E_i(z,s)$ associated to the i^{th} cusp of the group Γ is defined for $\operatorname{Re} s > 1$ by the formula

$$E_i(z,s) := \sum_{\gamma\in\Gamma_i\setminus\Gamma} \mathrm{Im}(\sigma_i^{-1}\gamma z)^s. \tag{8.27}$$

The Eisenstein-Mass series naturally descends to define a function on the quotient $\Sigma = \Gamma\setminus\mathbb{H}$. Recall that under the identification of $T_{[\Sigma]}T_{g,n}$ with the space of harmonic Beltrami differentials $\Omega^{-1,1}(\Sigma)$, the Weil-Peterson Kähler metric is defined by

$$\langle\mu,\nu\rangle_{WP} := \int_\Sigma \mu\overline{\nu}dg_\Sigma = \int_\Sigma \langle\mu,\nu\rangle_{K^{-1}\otimes\Lambda^{0,1}_\Sigma}dg_\Sigma \tag{8.28}$$

for $\mu,\nu \in T_{[\Sigma]}T_{g,n}$ with corresponding Kähler form given by

$$\omega_{WP}(\mu,\overline{\nu}) = \frac{\sqrt{-1}}{2}\langle\mu,\nu\rangle_{WP}. \tag{8.29}$$

To define their Kähler metric, Takhtajan and Zograf considered instead

$$\langle\mu,\nu\rangle_i = \int_\Sigma \mu\overline{\nu}E_i(\cdot,2)dg_\Sigma, \quad i=1,\ldots n. \tag{8.30}$$

Each of these scalar products gives rise to a Kähler metric on $T_{g,n}$ with corresponding Kähler form

$$\omega_{\mathrm{TZ},i}(\mu,\overline{\nu}) = \frac{\sqrt{-1}}{2}\langle \mu,\nu\rangle_i, \quad i=1,\ldots,n. \tag{8.31}$$

The sum of these metric is the Takhtajan-Zograf Kähler metric

$$\langle \mu,\nu\rangle_{\mathrm{TZ}} := \sum_{i=1}^{n} \langle \mu,\nu\rangle_i \tag{8.32}$$

with corresponding Kähler form given by

$$\omega_{\mathrm{TZ}}(\mu,\overline{\nu}) = \frac{\sqrt{-1}}{2}\langle \mu,\nu\rangle_{\mathrm{TZ}}. \tag{8.33}$$

We know from Corollary 8.5 that the eta form $\widehat{\eta}(D_\ell^V)_{[2]}$ is the Kähler form of a Kähler metric. This is consistent with Theorem 1 asserting that the eta form $\widehat{\eta}(D_\ell^V)$ is closed.

References

[1] L. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. Math. **74** (1961), 171–191.

[2] P. Albin and F. Rochon, *Family index for manifolds with hyperbolic cusp singularities*, to appear in IMRN.

[3] ———, *Some index formulae on the moduli space of stable parabolic vector bundles*, preprint, available at http://arxiv.org/abs/0812.2223.

[4] E. Arbarello and M. Cornalba, *Combinatorial and algebro geometric cohomology classes on the moduli space of curves*, J. Alg. Geom. **5** (1996), 705–749.

[5] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, Berlin, 1992.

[6] G. Bini, *Generalized Hodge classes on the moduli space of curve*, Contributions to Algebra and Geometry **44** (2003), no. 2, 559–565.

[7] J. M. Bismut and J. Cheeger, *η-invariants and their adiabatic limits*, J. Amer. Math. Soc. 2 **2** (1989), no. 1, 33–70.

[8] J. M. Bismut and D.S. Freed, *The analysis of elliptic families. I. metric and connections on the determinant bundles*, Commun. Math. Phys. **106** (1986), 159–176.

[9] J. M. Bismut, H. Gillet, and C. Soulé, *Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion*, Commun. Math. Phys. **115** (1988), 49–78.

[10] ———, *Analytic torsion and holomorphic determinant bundles. II. Direct images and Bott-Chern forms*, Commun. Math. Phys. **115** (1988), 79–126.

[11] ———, *Analytic torsion and holomorphic determinant bundles. III. Quillen metrics on holomorphic determinants*, Commun. Math. Phys. **115** (1988), 301–351.

[12] D. Borthwick, C. Judge, and P.A. Perry, *Selberg's zeta function and the spectral geometry of geometrically finite hyperbolic surfaces*, Comment. Math. Helv. **80** (2005), no. 3, 483–515.

[13] E. D'Hoker and D.H. Phong, *On determinants of Laplacians on Riemann surfaces*, Commun. Math. Phys. **104** (1986), 537–545.

[14] I. Efrat, *Determinants of Laplacians on surfaces of finite volume*, Commun. Math. Phys. **119** (1988), 443–451.

[15] ———, *Erratum: Determinants of Laplacians on surfaces of finite volume*, Commun. Math. Phys. **138** (1991), 607.

[16] P. Griffiths and J. Harris, *Principle of algebraic geometry*, Wiley, 1994.
[17] A. Hassell, *Analytic surgery and analytic torsion*, Comm. Anal. Geom. **6** (1998), no. 2, 255–289.
[18] J.H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics volume I: Teichmüller theory*, Matrix Editions, Ithaca NY, 2006.
[19] H.B. Lawson and M-L Michelsohn, *Spin geometry*, Princeton Univ. Press, Princeton, 1989.
[20] E. Leitchnam, R. Mazzeo, and P. Piazza, *The index of Dirac operators on manifolds with fibered boundaries*, To appear in the Bulletin of the Belgian Mathematical Society (2006).
[21] R. Mazzeo and R.B. Melrose, *Pseudodifferential operators on manifolds with fibred boundaries*, Asian J. Math. **2** (1998), no. 4, 833–866.
[22] R.B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research notes in mathematics, Wellesley, MA, 1993.
[23] ———, *The eta invariant and families of pseudodifferential operators*, Math. Res. Lett. **2** (1995), no. 5, 541–561.
[24] R.B. Melrose and P. Piazza, *Families of Dirac operators, boundaries and the b-calculus*, J. Differential Geom. **46** (1997), no. 1, 99–180. MR MR99a:58144
[25] ———, *An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary*, J. Differential Geom. **46** (1997), no. 2, 287–334. MR MR99a:58145
[26] W. Müller, *Spectral geometry and scattering theory for certain complete surfaces of finite volume*, Invent. math. **109** (1992), 265–305.
[27] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry Part II (M. Artin and J. Tate, eds), Birkhäuser (1983), 271–328.
[28] P. Piazza, *Determinant bundles, manifolds with boundary and surgery*, Commun. Math. Phys. **178** (1996), no. 3, 597–626.
[29] P. Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. **110** (1987), no. 1, 113–120.
[30] L.A. Takhtajan and P.G. Zograf, *A local index theorem for families of $\overline{\partial}$-operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces*, Comm. Math. Phys. **137** (1991), no. 2, 399–426.
[31] B. Vaillant, *Index and spectral theory for manifolds with generalized fibred cusps*, Ph.D. dissertation, Bonner Math. Schriften 344, Univ. Bonn., Mathematisches Institut, Bonn (2001), available online at arXiv: math/0102072v1.
[32] L. Weng, *Ω-admissible theory II. Deligne pairings over moduli spaces of punctured Riemann surfaces*, Math. Ann. **320** (2001), 239–283.
[33] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry **1** (1991), 243–310.
[34] S. Wolpert, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. math. **85** (1986), 119–145.
[35] ———, *Cusps and the family hyperbolic metric*, Duke Math. J. **138** (2007), no. 3, 423–443.
[36] W. Zhang, *Circle bundles, adiabatic limits of η-invariants and Rokhlin congruences*, Ann. Inst. Fourier **44** (1994), no. 1, 249–270.

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